

Correlation functions and representation bases in free $\mathcal{N} = 4$ Super Yang-Mills

Yusuke Kimura¹

Abstract

We study exact correlation functions of $\mathcal{N} = 4$ SYM at zero coupling. It has been known that it is convenient to label local gauge invariant operators by irreducible representations of symmetric groups/Brauer algebras. We first review the construction of representation bases from the viewpoint of the enhanced symmetry structure of the free theory. We present a basis of multi-matrix models using elements of Brauer algebras, generalising our previous construction for two matrices. We will compute multi-point functions of the basis with the exact N -dependence. In particular we study three-point functions of a class of BPS operators, and we find that they are given by a branching rule of the Brauer algebra. The three-point functions take a factorised form if representations on the operators satisfy a relation.

1 Introduction

During the last decade, the planar limit of $\mathcal{N} = 4$ Super Yang-Mills (SYM) has had a deep impact by the discovery of integrability, and the new structure has raised an expectation that $\mathcal{N} = 4$ SYM in the planar limit might be solved exactly. In contrast, we have a lack of knowledge about the non-planar theory. (See the papers [1, 2] for review.)

Study of $\mathcal{N} = 4$ SYM in the free field limit is a direction to focus on non-planar corrections. Even in the free theory, summing up all non-planar diagrams is a highly non-trivial issue, especially when our concern is operators with scaling dimensions of $\mathcal{O}(N)$ or $\mathcal{O}(N^2)$. Such operators are considered to be dual to giant gravitons [3, 4, 5] or general curved geometries [6]. One can be motivated to study $\mathcal{N} = 4$ SYM at zero coupling by recalling the fact that it is dual to a string theory defined on a highly curved space according to AdS/CFT. (In fact there is a conjecture that the free SYM is dual to a higher-spin gauge theory [7, 8, 9, 10].) Because the free field theory is much easier than the full theory, it would be a good lesson to study it in order to learn how to describe degrees of freedom corresponding to D-branes or geometries before going to the full interacting theory.

In this paper, we will compute exact correlation functions of local gauge invariant operators in $\mathcal{N} = 4$ SYM at zero coupling utilising the recently developed group theoretic technique. With

¹ Okayama Institute for Quantum Physics, Kyoyama 1-9-1, Kita-ku, Okayama, 700-0015, Japan; london-mileend@gmail.com

the help of group theory and representation theory, we have an efficient way to diagonalise two-point functions and obtain the exact N -dependence. The first result employing group theory was given in [11, 12] in which exact correlators in the 1/2 BPS sector was computed. Later it was generalised to include more fields in [13, 14, 15, 16, 17, 18]. These studies showed that operators that diagonalise two-point functions are labelled by representations of groups or algebras. We call such bases representation bases.

The organisation of this paper is as follows. In the next section we will explain how group representation theory can be an efficient tool in the free theory by reviewing the integrable structure of the free theory from the viewpoint of an enhanced symmetry given in [17]. In section 3, we will give an orthogonal basis of multi-matrix operators using the Brauer algebra, generalising our previous construction [13] for two matrices. We will also present explicit forms of the operator in some special sectors, which will be helpful to take a general meaning of representation labels on the operator. In section 4, we compute multi-point correlation functions of the Brauer operators. Section 5 is given for a study of correlation functions of the operators labelled by only irreducible representations of the Brauer algebra. Such operators were shown to be BPS in our previous paper [19]. We will see that the three-point functions of the BPS operators are given by a branching rule of the Brauer algebra. The Brauer basis would be suitable for constructing a composite from two kinds of operators. We will find that certain multi-point functions take a factorised form when representation labels of the operators satisfy a relation. In section 6 we will discuss future directions. In some appendices, we give some detailed computations, a brief review of representation theory, and the construction of commuting higher charges of the free system.

2 The free theory and higher conserved charges

In this section, we review a symmetry structure of $\mathcal{N} = 4$ SYM at zero coupling, and then we give an introduction of constructing representation bases by regarding the symmetry structure as a guiding principle.

Start from $\mathcal{N} = 4$ SYM theory defined on R^4 with $U(N)$ gauge group. For simplicity, we will discuss only the $SO(6)$ scalar sector, and the generalisation to the full sector will be mentioned in section 6. It is convenient to combine the six scalar fields into the complex combinations

$$X_1 = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad X_2 = \frac{1}{\sqrt{2}}(\Phi_3 + i\Phi_4), \quad X_3 = \frac{1}{\sqrt{2}}(\Phi_5 + i\Phi_6). \quad (1)$$

We also denote them by X, Y, Z .

The two-point functions take the following form

$$\langle O_i(x) O_j(y) \rangle_{SYM} = \frac{c_{ij}(N)}{(x-y)^{d_i+d_j}} \delta_{d_i, d_j}, \quad (2)$$

where d_i is the classical dimension of a gauge invariant operator O_i . General gauge invariant operators are given by a product of an arbitrary number of single trace operators built from the fundamental fields X_a and X_a^\dagger . At zero coupling, the scaling dimension is counted by the number of fields involved in the operator. The non-trivial N -dependence is encoded only in c_{ij} , which can be computed by the matrix integral with the Gaussian weight [20]

$$c_{ij}(N) = \langle O_i O_j \rangle := \int \prod_a [dX_a dX_a^\dagger] e^{-2\text{tr}(X_a X_a^\dagger)} O_i O_j, \quad (3)$$

where the measure is normalised to give $\langle X_j^i X_l^\dagger \rangle = \delta_l^i \delta_j^k$, and the fields are regarded as matrices without the space dependence. Throughout this paper, we do not focus on the space-time dependence because it is trivially recovered by the conformal invariance. In this way, the N -dependence of correlation functions in the free theory can be captured by the correlation functions of the matrix model. When an operator contains both X_a and X_a^\dagger , we define the operator by removing self-contractions. We introduce the normal ordering symbol $:O:$ in order to denote that it is free from self-contractions.

We next consider the radial quantisation of the theory by the conformal transformation from R^4 to $R \times S^3$. The action of the scalar fields obtains a mass term due to the conformal coupling to the curvature of S^3 . When our interest is the singlet sector under $SO(4)$ corresponding to the isometry of S^3 , we are left with the s-wave mode of the spherical harmonics of S^3 . The action is then given by the matrix quantum mechanics

$$S = \int dt \sum_a \text{tr} \left(\frac{1}{2} \dot{X}_a \dot{X}_a^\dagger - \frac{1}{2} X_a X_a^\dagger \right). \quad (4)$$

Defining annihilation and creation operators as we usually do, we get a collection of harmonic oscillators

$$H = \sum_a \text{tr}(A_a^\dagger A_a + B_a^\dagger B_a), \quad (5)$$

where we have ignored the zero-point energy. The Cartan elements of R-charge are given by

$$J_a = \text{tr}(-A_a^\dagger A_a + B_a^\dagger B_a). \quad (6)$$

By the radial quantisation, the dilatation operator defined on R^4 is mapped to the Hamiltonian on $R \times S^3$. The only non-trivial commutation relations are

$$[A_a^i{}_j, A_b^\dagger{}^k{}_l] = \delta_{ab} \delta_j^k \delta_l^i, \quad [B_a^i{}_j, B_b^\dagger{}^k{}_l] = \delta_{ab} \delta_j^k \delta_l^i. \quad (7)$$

The A -operators commute with the B -operators. States are built by acting with A_a^\dagger and B_a^\dagger on the vacuum uniquely determined by $A_a^i{}_j|0\rangle = 0$ and $B_a^i{}_j|0\rangle = 0$. Some descriptions of the matrix quantum mechanics in the 1/2 BPS sector were studied in [21].

The matrix integrals can be obtained by introducing the coherent state

$$|X, X^\dagger\rangle =: e^{\text{tr} X_a^\dagger B_a^\dagger} e^{\text{tr} X_a A_a^\dagger} : |0\rangle. \quad (8)$$

For gauge invariant states $|\Psi_i\rangle$, the inner product can be expressed by the matrix integral as

$$\langle \Psi_i | \Psi_j \rangle = \int \prod_a [dX_a dX_a^\dagger] e^{-2\text{tr}(X_a X_a^\dagger)} O_i(X, X^\dagger)^\dagger O_j(X, X^\dagger), \quad (9)$$

where O_i is the gauge invariant operator corresponding to a state $|\Psi_i\rangle$

$$O_i(X, X^\dagger) = \langle X, X^\dagger | \Psi_i \rangle. \quad (10)$$

We note again that operators are defined to be free from self-contractions in the integral. The matrix quantum mechanics of the non-holomorphic sector is also reviewed in [22] using a slightly different notation from here.

The system is a collection of the harmonic oscillators. Because it is an integrable system, it is expected to have a set of commuting conserved charges. We shall give a review of the construction of conserved charges based on [17]. In the course of the construction, group representation theory is naturally introduced, and higher charges will be simultaneously diagonalised by representation bases.

We now define

$$(G_{La}^B)^i{}_j := (B_a^\dagger B_a)^i{}_j, \quad (G_{Ra}^B)^i{}_j := -B_a^{\dagger k} B_a^i{}_k. \quad (11)$$

We find that

$$\begin{aligned} (G_{La}^B)^k{}_l B_b^{\dagger i}{}_j |0\rangle &= \delta_{ab} B_b^{\dagger k}{}_j \delta^i{}_l |0\rangle, \\ (G_{Ra}^B)^k{}_l B_b^{\dagger i}{}_j |0\rangle &= -\delta_{ab} B_b^{\dagger i}{}_l \delta^k{}_j |0\rangle. \end{aligned} \quad (12)$$

In the first equation, the lower index of the $(B^\dagger)^i{}_j$ is inert, while the upper index is inert in the second equation. The following commutators can be easily obtained from the basic commutator in (7)

$$\begin{aligned} [G_{La}^B, G_{Rb}^B] &= 0, \\ [(G_{sa}^B)^i{}_j, (G_{sb}^B)^k{}_l] &= \delta_{ab} ((G_{sa}^B)^i{}_l \delta_j^k - (G_{sa}^B)^k{}_j \delta_l^i) \quad (s = L, R). \end{aligned} \quad (13)$$

The above is six copies of $u(N)$ algebra. The usual adjoint $u(N)$ transformation is generated by $G_{La} + G_{Ra}$. The A -operators satisfy similar relations.

In terms of the generators, the Hamiltonian can be written by

$$H = \sum_a \text{tr}(G_{La}^A + G_{La}^B). \quad (14)$$

Because the Hamiltonian is a number operator, any operators built from G_{La} and G_{Ra} commute with the Hamiltonian. We call such operators conserved operators. We will build commuting conserved charges by regarding the pieces in (11) as fundamental building blocks.

We shall first consider the system in which states are excited by B_1^\dagger alone. It is the 1/2 BPS sector. We can find

$$[H_p, H_q] = 0, \quad H_p := \text{tr}((B_1^\dagger B_1)^p) = \text{tr}((G_{L1}^B)^p), \quad (15)$$

where p, q are positive integers. (See also [11, 23]). We have another sequence of conserved charges given by $\text{tr}((G_{R1}^B)^p)$, which have the same eigenvalues the operators $\text{tr}((G_{L1}^B)^p)$ have (see (C.2)).

The system we next consider is the system that contains B_a^\dagger ($a = 1, 2, 3$). In this case, the algebra of conserved operators is rather complicated, but we easily find that ²

$$\text{tr}((G_{L1}^B)^p), \quad \text{tr}((G_{L2}^B)^p), \quad \text{tr}((G_{L3}^B)^p) \quad (16)$$

commute each other. We also find that any conserved charges built from the pieces in (11) commute with the charges in (16). (The formula (B.2) can be available to very this.) Hence the charges in (16) are always included in a set of commuting conserved operators. We denote the set

² We also have $\text{tr}((G_{R1}^B)^p)$, $\text{tr}((G_{R2}^B)^p)$, $\text{tr}((G_{R3}^B)^p)$, which we do not show explicitly.

of the operators in (16) by S . In constructing commuting conserved charges, it is convenient to start from one of the following charges as well as S :

$$\begin{aligned} & \text{tr}((G_{L1}^B + G_{L2}^B + G_{L3}^B)^p), \quad \text{tr}((G_{L1}^B + G_{L2}^B + G_{R3}^B)^p), \quad \text{tr}((G_{R1}^B + G_{L2}^B + G_{R3}^B)^p), \\ & \text{tr}((G_{L1}^B + G_{R2}^B + G_{R3}^B)^p). \end{aligned} \quad (17)$$

Because they do not commute each other, we cannot put them together in a set. Based on one of the charges above with S , we can form a set of commuting conserved charges.

In what follows we will show how the basic conserved charges - S and (17) are diagonalised by representation theory. The basic ideas we shall exploit are as follows [17]. From (12), these higher charges are nothing but Casimir operators associated with the $u(N)$ actions. We will use the group theoretic fact that Casimirs are constant on an irreducible subspace of $V^{\otimes m} \otimes \bar{V}^{\otimes n}$, where V and \bar{V} are the fundamental representation and anti-fundamental representation of $U(N)$.

The construction of each set of commuting charges is studied more fully in appendix B, and the eigenvalues are shown in appendix C. Allowing excitations by A_a^\dagger is a straightforward extension.

We will construct a basis of gauge invariant states built from n_1 B_1^\dagger 's, n_2 B_2^\dagger 's, and n_3 B_3^\dagger 's. Start from the tensor product

$$B_1^{\dagger \otimes n_1} \otimes B_2^{\dagger \otimes n_2} \otimes B_3^{\dagger \otimes n_3} \quad (18)$$

as an endomorphism on $V^{\otimes n_1+n_2+n_3}$. It is known that the tensor product representation is reducible and can be decomposed as

$$V^{\otimes n_1+n_2+n_3} = \bigoplus_{R \vdash (n_1+n_2+n_3)} V_R^{U(N)} \otimes V_R^{S_{n_1+n_2+n_3}}. \quad (19)$$

The expression $R \vdash n$ indicates that R is summed over Young diagrams with n boxes. $V_R^{U(N)}$ and $V_R^{S_{n_1+n_2+n_3}}$ are the vector space of an irreducible representation R of $U(N)$ and that of $S_{n_1+n_2+n_3}$. The equation is known as Schur-Weyl duality. The Casimir operator $\text{tr}((G_{L1} + G_{L2} + G_{L3})^p)$ has a constant eigenvalue on the irreducible subspaces.

We can decompose the irreducible representation of $S_{n_1+n_2+n_3}$ into irreducible representations of the subgroup $S_{n_1} \times S_{n_2} \times S_{n_3}$ as

$$V_R^{S_{n_1+n_2+n_3}} = \bigoplus_{r_1 \vdash n_1, r_2 \vdash n_2, r_3 \vdash n_3} V_{R \rightarrow \vec{r}} \otimes V_{\vec{r}}^{S_{n_1} \times S_{n_2} \times S_{n_3}}, \quad (20)$$

where $\vec{r} := (r_1, r_2, r_3)$. The dimension of $V_{R \rightarrow \vec{r}}$ is given by the number of times the representation \vec{r} appears in the restriction of the representation R of $S_{n_1+n_2+n_3}$ to $S_{n_1} \times S_{n_2} \times S_{n_3}$, which is expressed by the Littlewood-Richardson coefficient as

$$\text{Dim}(V_{R \rightarrow \vec{r}}) = g(\vec{r}; R) := g(r_1, r_2, r_3; R). \quad (21)$$

Let us take a particular class of operators in the symmetric group $S_{n_1+n_2+n_3}$ that project on representations R, \vec{r} . We will denote it by $P_{\vec{r}, ij}^R$. The indices i, j are multiplicity indices running over $1, 2, \dots, g(\vec{r}; R)$. The important equations the operators satisfy are [15]

$$\begin{aligned} h P_{\vec{r}, ij}^R h^{-1} &= P_{\vec{r}, ij}^R \quad (h \in S_{n_1} \times S_{n_2} \times S_{n_3}), \\ P_{\vec{r}, ij}^R P_{\vec{r}', i' j'}^{R'} &= \delta^{RR'} \delta^{\vec{r} \vec{r}'} \delta_{ji'} P_{\vec{r}, ij}^R, \\ \text{tr}_{n_1+n_2+n_3}(P_{\vec{r}, ij}^R) &= d_{\vec{r}} \text{Dim} R \delta_{ij}, \end{aligned} \quad (22)$$

where $tr_{n_1+n_2+n_3}$ is a trace over the tensor product space (19). $d_{\vec{r}}$ is the dimension of an irreducible representation \vec{r} of $S_{n_1} \times S_{n_2} \times S_{n_3}$, and $Dim R$ is the dimension of an irreducible representation R of $U(N)$. Making use of the operator $P_{\vec{r},ij}^R$, we can build a gauge invariant state

$$|R, \vec{r}, ij\rangle := tr_{n_1+n_2+n_3}(P_{\vec{r},ij}^R B_1^{\dagger \otimes n_1} \otimes B_2^{\dagger \otimes n_2} \otimes B_3^{\dagger \otimes n_3})|0\rangle. \quad (23)$$

This basis is called restricted Schur basis. It forms an orthogonal basis [15, 24, 25]. It was originally introduced to describe open string excitations on giant gravitons [26, 27, 28]. For the action of the conserved charges on this basis, see Appendices B and C.

We next focus on another set of conserved charges that includes S and $tr((G_{L1} + G_{L2} + G_{R3})^p)$. This time it is convenient to start with the tensor product

$$B_1^{\dagger \otimes n_1} \otimes B_2^{\dagger \otimes n_2} \otimes B_3^{\dagger \otimes n_3}. \quad (24)$$

The tensor product representation can be decomposed as

$$V^{\otimes n_1+n_2} \otimes \bar{V}^{\otimes n_3} = \bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_N(n_1+n_2, n_3)}, \quad (25)$$

where γ runs over irreducible representations of the Brauer algebra $B_N(n_1 + n_2, n_3)$. The irreducible representations are labelled by a pair of Young diagrams (γ_+, γ_-) that have $n_1 + n_2 - k$ and $n_3 - k$ boxes, where k is an integer that satisfies $0 \leq k \leq \min(n_1 + n_2, n_3)$. We also denote them by (γ_+, γ_-, k) to place an importance on the value of k , because the integer k plays an important role in this paper. The representation theory is briefly summarised in appendix A. The Brauer algebra plays the same role the symmetric group does in Schur-Weyl duality (19).

Because the Brauer algebra contains the group algebra $\mathbb{C}[S_{n_1} \times S_{n_2} \times S_{n_3}]$ as a subalgebra, we have

$$V_{\gamma}^{B_N(n_1+n_2, n_3)} = \bigoplus_{r_1 \vdash n_1, r_2 \vdash n_2, r_3 \vdash n_3} V_{\gamma \rightarrow r} \otimes V_{\vec{r}}^{\mathbb{C}[S_{n_1} \times S_{n_2} \times S_{n_3}]}, \quad (26)$$

where $V_{\gamma \rightarrow \vec{r}}$ is a vector space associated with the restriction, whose dimension is given by ³

$$M_{\vec{r}}^{\gamma} = Dim(V_{\gamma \rightarrow \vec{r}}) = \sum_{t \vdash (n_1+n_2)} \sum_{\tau \vdash k} g(\gamma_-, \tau; r_3) g(\gamma_+, \tau; t) g(r_1, r_2; t). \quad (29)$$

³ It can be derived by decomposing the multiplicity as

$$M_{\vec{r}}^{\gamma} = \sum_{t \vdash (n_1+n_2)} M_{(t, r_3)}^{\gamma} M_{(r_1, r_2)}^t, \quad (27)$$

where the two factors in the RHS come from the restrictions $B_N(n_1+n_2, n_3) \rightarrow \mathbb{C}[S_{n_1+n_2}] \times \mathbb{C}[S_{n_3}]$ and $\mathbb{C}[S_{n_1+n_2}] \rightarrow \mathbb{C}[S_{n_1}] \times \mathbb{C}[S_{n_2}]$. The multiplicity of the first restriction is given by the form (82). We have another way to decompose the multiplicity

$$M_{\vec{r}}^{\gamma} = \sum_{\gamma_1} M_{(\gamma_1, r_2)}^{\gamma} M_{(r_1, r_3)}^{\gamma_1}, \quad (28)$$

where γ_1 is an irreducible representation of the $B_N(n_1, n_3)$. The two restrictions are $B_N(n_1+n_2, n_3) \rightarrow B_N(n_1, n_3) \times B_N(n_2, 0)$ and $B_N(n_1, n_3) \rightarrow \mathbb{C}[S_{n_1}] \times \mathbb{C}[S_{n_3}]$. The multiplicity associated with the first one is referred to (76) and (86).

We now introduce particular linear combinations of elements in the Brauer algebra that are associated with representations γ, \vec{r} . We denote it by $Q_{\vec{r}, ij}^\gamma$. (More information about it will be given in the next section.) Making use of this operator, we construct a gauge invariant state as

$$|\gamma, \vec{r}, ij\rangle := tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^\gamma B_1^{\dagger \otimes n_1} \otimes B_2^{\dagger \otimes n_2} \otimes B_3^{\dagger T \otimes n_3})|0\rangle, \quad (30)$$

where the trace $tr_{n_1+n_2, n_3}$ is taken over the space (25). We call this basis Brauer basis or basis based on the Brauer algebra $B_N(n_1 + n_2, n_3)$. When $n_2 = 0$, it is the basis proposed in [13, 17]. In the next section we will supplement this basis with some properties. The conserved charges are examined more fully in Appendices B and C.

It is also possible to consider in stead of (24)

$$B_1^{\dagger \otimes n_1} \otimes B_2^{\dagger T \otimes n_2} \otimes B_3^{\dagger \otimes n_3} \quad (31)$$

or

$$B_1^{\dagger T \otimes n_1} \otimes B_2^{\dagger \otimes n_2} \otimes B_3^{\dagger \otimes n_3}, \quad (32)$$

where $B_N(n_1 + n_3, n_2)$ or $B_N(n_2 + n_3, n_1)$ plays the role respectively. These are related to the conserved charges $tr((G_{L1} + G_{R2} + G_{L3})^p)$ or $tr((G_{R1} + G_{L2} + G_{L3})^p)$. These are completely similar to the case in which $B_N(n_1 + n_2, n_3)$ plays the role.

We now give an expression of the conserved charges as differential operators on the matrices X, Y , etc. The operators in the set S are mapped to

$$tr((X\partial_X)^p), \quad tr((Y\partial_Y)^p), \quad tr((Z\partial_Z)^p), \quad (33)$$

while the operators $tr((G_{L1}^B + G_{L2}^B + G_{L3}^B)^p)$ and $tr((G_{L1}^B + G_{L2}^B + G_{R3}^B)^p)$ are mapped to

$$tr((X\partial_X + Y\partial_Y + Z\partial_Z)^p), \quad tr((X\partial_X + Y\partial_Y - : \partial_Z Z :)^p), \quad (34)$$

where $:(\partial_Z Z)_{ij} := Z_{kj}(\partial_Z)_{ik}$. We also have $tr((G_{R1}^A)^p) \rightarrow tr((:\partial_{\bar{X}} \bar{X}:)^p)$ etc.

Before moving to the next section, mentioned is the use of (11) as a building-block of conserved quantities. It was possible to add a more general conserved building block like $(B_a^\dagger)_{ij}(B_a)_{kl}$ in the set of building blocks. When we think about states excited by only $(B_1^\dagger)_{ij}$, we only get the same set as (15). On the other hand, when we have more than one matrix, we can construct a different set of conserved operators if we are allowed to use $(B_a^\dagger)_{ij}(B_a)_{kl}$ in addition to (11). In fact, it was shown in [17] that the building block $(B_a^\dagger)_{ij}(B_a)_{kl}$ is needed to construct conserved charges to measure the labels of the basis given in [14, 16]. We leave it as a future homework to give a complete classification of conserved charges by beginning with more general building blocks.

3 Orthogonal basis for multi-matrix using Brauer algebra

In this section, we will study the representation basis (30) in more detail. This basis itself is new, but most properties are generalisations of the Brauer basis for two matrices given in [13, 17, 22].

Recall the basis

$$O_{\vec{r}, ij}^\gamma(X, Y, Z) := \langle X, Y, Z | \gamma, \vec{r}, ij \rangle = tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^\gamma X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{T \otimes n_3}). \quad (35)$$

The operator $Q_{\vec{r},ij}^\gamma$ is a linear combination of elements in the Brauer algebra [13, 17],

$$Q_{\vec{r},ij}^\gamma = \text{Dim}\gamma \sum_{b \in B_N(n_1+n_2, n_3)} \chi_{\vec{r},ji}^\gamma(b) b^*. \quad (36)$$

Here we have defined the restricted character⁴ of a representation γ of the Brauer algebra

$$\chi_{\vec{r},ij}^\gamma(b) := \sum_{m=1}^{d_{\vec{r}}} \langle \gamma; \vec{r}, m, i | b | \gamma; \vec{r}, m, i \rangle \quad (37)$$

where $|\gamma; \vec{r}, m, i\rangle$ is an orthonormal vector in the RHS of (26). We will not need the explicit form of $Q_{\vec{r},ij}^\gamma$ hereafter in this paper, but we will often use the following important properties

$$\begin{aligned} h Q_{\vec{r},ij}^\gamma h^{-1} &= Q_{\vec{r},ij}^\gamma \quad (h \in S_{n_1} \times S_{n_2} \times S_{n_3}), \\ Q_{\vec{r},ij}^\gamma Q_{\vec{r}',i'j'}^{\gamma'} &= \delta^{\gamma\gamma'} \delta^{\vec{r}\vec{r}'} \delta_{ji'} Q_{\vec{r},ij}^\gamma, \\ \text{tr}_{n_1+n_2, n_3}(Q_{\vec{r},ij}^\gamma) &= d_{\vec{r}} \text{Dim}\gamma \delta_{ij}. \end{aligned} \quad (38)$$

$d_{\vec{r}}$ is the dimension of an irreducible representation \vec{r} of $S_{n_1} \times S_{n_2} \times S_{n_3}$, and $\text{Dim}\gamma$ is the dimension of an irreducible representation γ of $U(N)$. The third equation is a consequence of the Schur-Weyl duality (25).

We will now show that two-point functions of the basis are diagonal. The two-point functions can be computed with the exact N -dependence

$$\begin{aligned} \langle \gamma', \vec{r}', i' j' | \gamma, \vec{r}, i j \rangle &= \sum_{h \in H} \text{tr}_{n_1+n_2, n_3}(Q_{\vec{r},ij}^\gamma h Q_{\vec{r}',i'j'}^{\gamma'} h^{-1}) \\ &= n_1! n_2! n_3! \text{tr}_{n_1+n_2, n_3}(Q_{\vec{r},ij}^\gamma Q_{\vec{r}',i'j'}^{\gamma'}) \\ &= n_1! n_2! n_3! \delta_{\gamma\gamma'} \delta_{\vec{r}\vec{r}'} \delta_{ii'} \delta_{jj'} d_{\vec{r}} \text{Dim}\gamma, \end{aligned} \quad (39)$$

where $H = S_{n_1} \times S_{n_2} \times S_{n_3}$. The N -dependence is contained in $\text{Dim}\gamma$. To get the first equality, we have exploited the fact that the Wick-contractions can be easily performed with the help of the symmetric group [11]. The formula we have used is as follows,

$$\langle X_{j_1}^{i_1} \cdots X_{j_n}^{i_n} X_{l_1}^{\dagger k_1} \cdots X_{l_n}^{\dagger k_n} \rangle = \sum_{\sigma \in S_n} \left(\delta_{l_{\sigma(1)}}^{i_1} \delta_{j_{\sigma^{-1}(1)}}^{k_1} \right) \left(\delta_{l_{\sigma(2)}}^{i_2} \delta_{j_{\sigma^{-1}(2)}}^{k_2} \right) \cdots \left(\delta_{l_{\sigma(n)}}^{i_n} \delta_{j_{\sigma^{-1}(n)}}^{k_n} \right). \quad (40)$$

Because we have three matrices X, Y, Z , the Wick-contractions are given by the elements in H . Likewise we have

$$\langle 0 | B_{j_1}^{i_1} \cdots B_{j_n}^{i_n} B_{l_1}^{\dagger k_1} \cdots B_{l_n}^{\dagger k_n} | 0 \rangle = \sum_{\sigma \in S_n} \left(\delta_{l_{\sigma(1)}}^{i_1} \delta_{j_{\sigma^{-1}(1)}}^{k_1} \right) \left(\delta_{l_{\sigma(2)}}^{i_2} \delta_{j_{\sigma^{-1}(2)}}^{k_2} \right) \cdots \left(\delta_{l_{\sigma(n)}}^{i_n} \delta_{j_{\sigma^{-1}(n)}}^{k_n} \right). \quad (41)$$

For an element b in the Brauer algebra $B_N(n_1 + n_2, n_3)$, we have the formula

$$\frac{1}{n_1! n_2! n_3!} \sum_{h \in H} h b h^{-1} = \sum_{\gamma, \vec{r}, ij} \frac{1}{d_{\vec{r}}} \chi_{\vec{r},ij}^\gamma(b) Q_{\vec{r},ij}^\gamma. \quad (42)$$

⁴ If we take the sum of \vec{r} and i after setting $i = j$, it is the character of a representation γ of the Brauer algebra.

Both sides of (42) commute with any elements in H . The formula leads to the following equation

$$tr_{n_1+n_2, n_3}(bX \otimes Y \otimes Z^T) = \sum_{\gamma, r, ij} \frac{1}{d_r} \chi_{r, ij}^\gamma(b) tr_{n_1+n_2, n_3}(Q_{r, ij}^\gamma X \otimes Y \otimes Z^T). \quad (43)$$

Note that there are equivalence classes under conjugation of the group action of H .⁵ Taking into account that any gauge invariant operator built from X 's, Y 's, and Z 's can be expressed in terms of an element of the Brauer algebra in the form of the left-hand side of the above equation, the above equation means the completeness of the basis.

When we have $N \times N$ matrices, we always have some relations between multi-traces. A simple example is $tr(\phi^3) = 3/2 tr(\phi)tr(\phi^2) - 1/2(tr\phi)^3$ for a 2×2 matrix ϕ . In general, $tr(\phi^p)$ ($p > N$) can be expressed by a linear combination of $tr(\phi^q)$ ($q \leq N$). An advantage of using representation bases is that finite N relations are more manifest because they are expressed by constraints on the Young diagrams. For the Brauer basis, we have constraints

$$\begin{aligned} c_1(\gamma_+) + c_1(\gamma_-) &\leq N, \\ c_1(r_1) &\leq N, \quad c_1(r_2) \leq N, \quad c_1(r_3) \leq N, \end{aligned} \quad (44)$$

where $c_1(r)$ represents the first column length of a Young diagram r .

Hereafter we will present concrete forms of the operator at two sectors in which k takes the possible maximum value and the possible minimum value. This will enable us to guess the meaning of the integer k . The derivations being completely analogous to what we did in [13, 17, 22], we will only show the final forms with leaving concrete derivations in appendix D.

In the $k = 0$ sector, γ_- and r_3 are identified, which we find from (29). The $k = 0$ operators take the form

$$\begin{aligned} &tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^{\gamma(k=0)} X \otimes Y \otimes Z^T) \\ &= Dim\gamma \frac{(n_1 + n_2)! n_3!}{d_{\gamma_+}} \frac{1}{d_{r_3}} \frac{1}{N^n} tr_n(\Omega_n^{-1} P_{(r_1, r_2), ij}^{\gamma_+} p_{\gamma_-} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3}), \end{aligned} \quad (45)$$

where $n := n_1 + n_2 + n_3$. Note that Z 's are not transposed in the second line. Because Ω_n^{-1}/N^n starts from 1 in a $1/N$ -expansion, the leading term with respect to $1/N$ is the product of a restricted Schur polynomial built from $P_{(r_1, r_2), ij}^{\gamma_+}$ and a Schur polynomial built from p_{γ_-} :

$$\begin{aligned} &\frac{1}{N^n} tr_n(\Omega_n^{-1} P_{(r_1, r_2), ij}^{\gamma_+} p_{\gamma_-} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3}) \\ &= tr_{n_1+n_2}(P_{(r_1, r_2), ij}^{\gamma_+} X^{\otimes n_1} \otimes Y^{\otimes n_2}) tr_{n_3}(p_{\gamma_-} Z^{\otimes n_3}) + \dots \end{aligned} \quad (46)$$

This implies that the Brauer basis is suitable for organising a composite of two operators.

Because the above expression (45) is written in terms of the tensor product appeared in the construction of the restricted Schur basis, we can easily express (45) as a linear combination of the restricted Schur operators

$$\begin{aligned} &tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^{\gamma(k=0)} X \otimes Y \otimes Z^T) = \\ &Dim\gamma \frac{(n_1 + n_2)! n_3!}{n!} \frac{1}{d_{\gamma_+} d_{r_3}} \sum_{S \vdash n, k, l} \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, kl}^S \left(P_{(r_1, r_2), ij}^{\gamma_+} \right) \frac{d_S}{Dim_S} tr_n(P_{\vec{r}, kl}^S X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3}). \end{aligned} \quad (47)$$

⁵ We have $tr_{n_1+n_2, n_3}(bX \otimes Y \otimes Z^T) = tr_{n_1+n_2, n_3}(b'X \otimes Y \otimes Z^T)$ if $b' = h b h^{-1}$. Note also $\chi_{r, ij}^\gamma(b) = \chi_{r, ij}^\gamma(b')$.

The two operators have the same representation \vec{r} .

We next look at the case in which k takes the maximum possible value. Suppose $n_1 + n_2 = n_3$, for simplicity. When k takes the maximum value $k = n_1 + n_2 = n_3$, γ is (\emptyset, \emptyset) , and the multiplicity labels i, j run over $1, \dots, g(r_1, r_2; r_3)$. The explicit form in this sector is given by

$$\begin{aligned} & tr_{n_1+n_2, n_3} (Q_{(r_1, r_2, r_3), ij}^{\gamma(k=n_1+n_2=n_3)} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{T \otimes n_3}) \\ &= \frac{d_{r_3}}{Dim r_3} tr_{n_1+n_2} (P_{(r_1, r_2), ij}^{r_3} (ZX)^{\otimes n_1} \otimes (ZY)^{\otimes n_2}). \end{aligned} \quad (48)$$

This is nothing but a restricted Schur operator built from two matrices ZX and ZY . Hence the operator (48) looks describing open string excitations on giant gravitons where their angular momenta are excited along some directions. This sector might be helpful to describe a configuration involving such combined matrices.

We have seen the two sectors of the Brauer basis, where k takes the maximum possible value and the minimum possible value. From the two examples, we may be led to an interpretation of k that it corresponds a degree of freedom describing the mixing between the X - Y sector and the Z sector. In [29] the integer k was given a meaning in the context of quarter BPS bubbling geometries, where it was conjectured that the mixing between the two angular directions of the geometries is labelled by k .

Let us now comment on including anti-holomorphic matrices. Originally we guessed that the Brauer basis would be a good basis for describing a system that contains giant gravitons and anti-giant gravitons [13]. With the same spirit, describing a more general system of giant gravitons and anti-giant gravitons is naturally given by considering the Brauer algebra $B_N(n_1 + n_2 + n_3, \bar{n}_1 + \bar{n}_2 + \bar{n}_3)$

$$tr_{n_1+n_2+n_3, \bar{n}_1+\bar{n}_2+\bar{n}_3} (Q_{\vec{r}, ij}^{\gamma} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3} \otimes X^{*\otimes \bar{n}_1} \otimes Y^{*\otimes \bar{n}_2} \otimes Z^{*\otimes \bar{n}_3}), \quad (49)$$

where $\vec{r} = (s_1, s_2, s_3, t_1, t_2, t_3)$ is an irreducible representation of $\mathbb{C}[S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{\bar{n}_1} \times S_{\bar{n}_2} \times S_{\bar{n}_3}]$. The γ is an irreducible representation of the Brauer algebra; $\gamma_+ \vdash (n_1 + n_2 + n_3 - k)$, $\gamma_- \vdash (\bar{n}_1 + \bar{n}_2 + \bar{n}_3 - k)$, $0 \leq k \leq \min(n_1 + n_2 + n_3, \bar{n}_1 + \bar{n}_2 + \bar{n}_3)$. The leading term of the $k = 0$ operators is indeed a product of two operators corresponding to the holomorphic sector and the anti-holomorphic sector, which can be written up to a numerical factor as

$$tr_{n_1+n_2+n_3} (P_{\vec{s}, kl}^{\gamma_+} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3}) tr_{\bar{n}_1+\bar{n}_2+\bar{n}_3} (P_{\vec{t}, pq}^{\gamma_-} X^{\dagger \otimes \bar{n}_1} \otimes Y^{\dagger \otimes \bar{n}_2} \otimes Z^{\dagger \otimes \bar{n}_3}). \quad (50)$$

We also have an option to utilise the symmetric group $S_{n_1+n_2+n_3+\bar{n}_1+\bar{n}_2+\bar{n}_3}$ [15]

$$tr_{n_1+n_2+n_3, \bar{n}_1+\bar{n}_2+\bar{n}_3} (P_{\vec{r}, ij}^R X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3} \otimes X^{\dagger \otimes \bar{n}_1} \otimes Y^{\dagger \otimes \bar{n}_2} \otimes Z^{\dagger \otimes \bar{n}_3}). \quad (51)$$

It is also possible to use another Brauer algebra such as $B_N(n_1 + n_2 + n_3 + \bar{n}_1, \bar{n}_2 + \bar{n}_3)$

$$tr_{n_1+n_2+n_3+\bar{n}_1, \bar{n}_2+\bar{n}_3} (Q_{\vec{r}, ij}^{\gamma'_+} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{\otimes n_3} \otimes X^{\dagger \otimes \bar{n}_1} \otimes Y^{*\otimes \bar{n}_2} \otimes Z^{*\otimes \bar{n}_3}), \quad (52)$$

where $\gamma'_+ \vdash (n_1 + n_2 + n_3 + \bar{n}_1 - k)$ and $\gamma'_- \vdash (\bar{n}_2 + \bar{n}_3 - k)$, $0 \leq k \leq \min(n_1 + n_2 + n_3 + \bar{n}_1, \bar{n}_2 + \bar{n}_3)$. Another kind of basis of non-holomorphic operators was given in [18] by combining a Brauer algebra and the method in [14, 16].

4 Multi-point correlation functions

In this section, we will be concerned with multi-point correlation functions of the Brauer operators. Multi-point functions can be computed efficiently by a product rule [24], which allows us to carry out the computation of multi-point functions in terms of the two-point function we obtained in (39).

We first consider the holomorphic operators. The product rule of this case is as follows

$$\begin{aligned}
& tr_{n_{X1}+n_{Y1},n_{Z1}}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1} X^{\otimes n_{X1}} \otimes Y^{\otimes n_{Y1}} \otimes Z^{T \otimes n_{Z1}}) \\
& \times tr_{n_{X2}+n_{Y2},n_{Z2}}(Q_{\vec{r}_2,i_2j_2}^{\gamma_2} X^{\otimes n_{X2}} \otimes Y^{\otimes n_{Y2}} \otimes Z^{T \otimes n_{Z2}}) \\
& = tr_{n_X+n_Y,n_Z}((Q_{\vec{r}_1,i_1j_1}^{\gamma_1} \circ Q_{\vec{r}_2,i_2j_2}^{\gamma_2}) X^{\otimes n_X} \otimes Y^{\otimes n_Y} \otimes Z^{T \otimes n_Z}) \\
& = \sum_{\gamma,\vec{r},ij} \frac{1}{d_{\vec{r}}} \chi_{\vec{r},ij}^{\gamma} (Q_{\vec{r}_1,i_1j_1}^{\gamma_1} \circ Q_{\vec{r}_2,i_2j_2}^{\gamma_2}) tr_{n_X+n_Y,n_Z}(Q_{\vec{r},ij}^{\gamma} X^{\otimes n_X} \otimes Y^{\otimes n_Y} \otimes Z^{T \otimes n_Z}), \quad (53)
\end{aligned}$$

where $n_X = n_{X1} + n_{X2}$, $n_Y = n_{Y1} + n_{Y2}$, $n_Z = n_{Z1} + n_{Z2}$. To get the last equality we have exploited (43). Thanks to the product rule, we immediately obtain the following expression for three-point functions

$$\langle O_{\vec{r}_1,i_1j_1}^{\gamma_1} O_{\vec{r}_2,i_2j_2}^{\gamma_2} O_{\vec{r},ij}^{\gamma \dagger} \rangle = n_X! n_Y! n_Z! Dim \gamma \chi_{\vec{r},ij}^{\gamma} (Q_{\vec{r}_1,i_1j_1}^{\gamma_1} \circ Q_{\vec{r}_2,i_2j_2}^{\gamma_2}). \quad (54)$$

Note that the expectation value is evaluated by the matrix model. For reference, we write the two-point function in a similar form

$$\langle O_{\vec{r}_1,i_1j_1}^{\gamma_1} O_{\vec{r}_2,i_2j_2}^{\gamma_2 \dagger} \rangle = n_X! n_Y! n_Z! Dim \gamma_2 \chi_{\vec{r}_2,i_2j_2}^{\gamma_2} (Q_{\vec{r}_1,i_1j_1}^{\gamma_1}). \quad (55)$$

The result can also be extended to multi-point functions

$$\begin{aligned}
& \langle O_{\vec{r}_1,i_1j_1}^{\gamma_1} \cdots O_{\vec{r}_s,i_sj_s}^{\gamma_s} O_{\vec{r}_{s+1},i_{s+1}j_{s+1}}^{\gamma_{s+1} \dagger} \cdots O_{\vec{r}_t,i_tj_t}^{\gamma_t \dagger} \rangle \\
& = n_X! n_Y! n_Z! \sum_{\gamma,\vec{r},ij} Dim \gamma \frac{1}{d_{\vec{r}}} \chi_{\vec{r},ij}^{\gamma} (Q_{\vec{r}_1,i_1j_1}^{\gamma_1} \circ \cdots \circ Q_{\vec{r}_s,i_sj_s}^{\gamma_s}) \chi_{\vec{r},ji}^{\gamma} (Q_{\vec{r}_{s+1},i_{s+1}j_{s+1}}^{\gamma_{s+1}} \circ \cdots \circ Q_{\vec{r}_t,i_tj_t}^{\gamma_t}), \quad (56)
\end{aligned}$$

where $n_X = n_{X1} + \cdots + n_{Xs} = n_{X_{s+1}} + \cdots + n_{Xt}$, and n_Y and n_Z are also similarly defined. For comparison, we recall the result on the restricted Schur basis (in our notation) [24]

$$\langle O_{\vec{r}_1,i_1j_1}^{R_1} O_{\vec{r}_2,i_2j_2}^{R_2} O_{\vec{r},ij}^{R \dagger} \rangle = n_X! n_Y! n_Z! Dim R \chi_{\vec{r},ij}^R (P_{\vec{r}_1,i_1j_1}^{R_1} \circ P_{\vec{r}_2,i_2j_2}^{R_2}). \quad (57)$$

The character was concretely evaluated for some cases in [24].

In this way, the evaluation of correlation functions has been replaced with the evaluation of the characters of symmetric groups or Brauer algebras.

We will now provide more concrete forms of the correlators for operators in the two specific sectors. Let us consider a correlator of the $k = 0$ operators. Recall that the $k = 0$ representations of the Brauer algebra are factorised to be

$$V_{\gamma(k=0)}^{B_N(n_1+n_2,n_3)} = V_{\gamma_+}^{C[S_{n_1+n_2}]} \otimes V_{\gamma_-}^{C[S_{n_3}]}. \quad (58)$$

This means

$$\chi^{\gamma(k=0)}(b) = 0 \quad (59)$$

for b that is not an element in the subalgebra $\mathbb{C}[S_{n_1+n_2}] \times \mathbb{C}[S_{n_3}]$. This forces k_1, k_2 to be zero in order to get a non-zero correlator. This property allows us to rewrite the character as a form where γ_+ -part and the γ_- -part are factored

$$\chi_{\vec{r},ij}^{\gamma(k=0)}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1(k=0)} \circ Q_{\vec{r}_2,i_2j_2}^{\gamma_2(k=0)}) = \chi_{(r_1,r_2),ij}^{\gamma_+}(P_{(r_{11},r_{12}),i_1j_1}^{\gamma_1+} \circ P_{(r_{21},r_{22}),i_2j_2}^{\gamma_2+}) d_{r_{13}} d_{r_{23}} g(r_{13}, r_{23}; r_3), \quad (60)$$

where $\vec{r}_1 = (r_{11}, r_{12}, r_{13})$, $\vec{r}_2 = (r_{21}, r_{22}, r_{23})$. The first factor in the right-hand side is the factor appearing in the three-point function of the restricted Schur-polynomials built from two matrices X, Y (see (57)), while the second factor is a factor appearing in the three-point function of the Schur-polynomials built from Z (see [11]). In the next section, we will give a systematic study of a condition a class of correlators are factorised.

When all k 's have the possible maximum values with conditions $n_X + n_Y = n_Z$, $n_{X1} + n_{Y1} = n_{Z1}$, and $n_{X2} + n_{Y2} = n_{Z2}$, that is, all operators are expressed by the form (48), we have

$$\begin{aligned} & \langle O_{\vec{r}_1,i_1j_1}^{\gamma_1(k_1=n_{Z1})} O_{\vec{r}_2,i_2j_2}^{\gamma_2(k_2=n_{Z2})} O_{\vec{r},ij}^{\gamma(k=n_Z)\dagger} \rangle \\ &= n_X! n_Y! n_Z! \frac{Dim r_3}{Dim r_1 Dim r_2} d_{r_1} d_{r_2} \chi_{(r_1,r_2),ij}^{r_3}(P_{(r_{11},r_{12}),i_1j_1}^{r_{13}} \circ P_{(r_{21},r_{22}),i_2j_2}^{r_{23}}). \end{aligned} \quad (61)$$

It is non-zero if $k_1 + k_2 = k$, which comes from R-charge conservation. From (57), the above character factor implies that the correlator is equivalent to a correlator of the restricted Schur operators built from two matrices.

From now onwards we include non-holomorphic operators. In order to illustrate how to compute such correlation functions, we will consider the following as a simple example

$$\langle : tr_{m_1,n_1}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1} X^{\otimes m_1} \otimes X^{*\otimes n_1}) :: tr_{m_2,n_2}(Q_{\vec{r}_2,i_2j_2}^{\gamma_2} X^{\otimes m_2} \otimes X^{*\otimes n_2}) :: tr_{m,n}(Q_{\vec{r},ji}^{\gamma} X^{\dagger \otimes m} \otimes X^{T \otimes n}) : \rangle. \quad (62)$$

From R-charge conservation, it is zero unless $m_1 + m_2 + n = n_1 + n_2 + m$ is satisfied.

If we have $m = m_1 + m_2$ and $n = n_1 + n_2$, X^\dagger 's (or X^T 's) in the third operator have to be contracted with X 's (or X^* 's) in the first one and X 's (or X^* 's) in the second one, and we do not have any contractions between the first operator and the second operator. We are then allowed to use the product rule (53) to get a similar result to (54). But if $m = m_1 + m_2$, $n = n_1 + n_2$ are not satisfied, Wick-contractions between the first operator and the second operator are missed if the product rule (53) is naively applied to this case.

The product rule for the case under consideration is given by

$$\begin{aligned} & : tr_{m_1,n_1}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1} X^{\otimes m_1} \otimes X^{*\otimes n_1}) :: tr_{m_2,n_2}(Q_{\vec{r}_2,i_2j_2}^{\gamma_2} X^{\otimes m_2} \otimes X^{*\otimes n_2}) : \\ &= e^{tr(\partial_X \partial_{Y^\dagger}) + tr(\partial_Y \partial_{X^\dagger})} : tr_{m_1,n_1}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1} X^{\otimes m_1} \otimes X^{*\otimes n_1}) tr_{m_2,n_2}(Q_{\vec{r}_2,i_2j_2}^{\gamma_2} Y^{\otimes m_2} \otimes Y^{*\otimes n_2}) : \begin{matrix} [Y=X] \\ [Y^\dagger=X^\dagger] \end{matrix}. \end{aligned}$$

The symbol $[Y = X], [Y^\dagger = X^\dagger]$ appearing at the lower right of the equation means we set $Y = X, Y^\dagger = X^\dagger$ after performing the exponential operation. This is Wick's theorem.

In order to see the effect of the exponential factor, let us consider

$$\begin{aligned} & tr(\partial_X \partial_{Y^\dagger}) : tr_{m_1,n_1}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1} X^{\otimes m_1} \otimes X^{*\otimes n_1}) tr_{m_2,n_2}(Q_{\vec{r}_2,i_2j_2}^{\gamma_2} Y^{\otimes m_2} \otimes Y^{*\otimes n_2}) : [Y=X, Y^\dagger=X^\dagger] \\ &= \sum_{1 \leq a \leq m_1, n_1+1 \leq b \leq n_1+n_2} : tr_{m_1+m_2, n_1+n_2}(C_{ab}(Q_{\vec{r}_1,i_1j_1}^{\gamma_1} \circ Q_{\vec{r}_2,i_2j_2}^{\gamma_2}) \\ & \quad \times X^{\otimes a-1} \otimes 1 \otimes X^{m_1+m_2-a} \otimes X^{*\otimes b-1} \otimes 1 \otimes X^{*\otimes n_1+n_2-b}) : , \end{aligned} \quad (63)$$

where C_{ab} is the contraction operator acting on the a -th X and the b -th X^*

$$tr(\partial_X \partial_{X^\dagger}) X_j^i X_k^{*l} = \delta^{il} \delta_{jk} = (C_{ab})_{jk}^{il}. \quad (64)$$

The action of C_{ab} ($1 \leq a \leq m_1, n_1 + 1 \leq b \leq n_1 + n_2$) on $(Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2})$ is explicitly written down as

$$\begin{aligned} C_{ab} &: (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2})_{j_1 \dots j_a \dots j_{m_1} j_{m_1+1} \dots j_{m_1+n_1}; l_1 \dots l_{m_2} l_{m_2+1} \dots l_b \dots l_{m_2+n_2}}^{i_1 \dots i_a \dots i_{m_1} i_{m_1+1} \dots i_{m_1+n_1}; k_1 \dots k_{m_2} k_{m_2+1} \dots k_b \dots k_{m_2+n_2}} \\ &\rightarrow \delta_{i_a k_b} \sum_s (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2})_{j_1 \dots j_a \dots j_{m_1} j_{m_1+1} \dots j_{m_1+n_1}; l_1 \dots l_{m_2} l_{m_2+1} \dots l_b \dots l_{m_2+n_2}}^{i_1 \dots s \dots i_{m_1} i_{m_1+1} \dots i_{m_1+n_1}; k_1 \dots k_{m_2} k_{m_2+1} \dots s \dots k_{m_2+n_2}}. \end{aligned} \quad (65)$$

Similarly we have

$$\begin{aligned} &tr(\partial_Y \partial_{Y^\dagger}) : tr_{m_1, n_1} (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} X^{\otimes m_1} \otimes X^{*\otimes n_1}) tr_{m_2, n_2} (Q_{\vec{r}_2, i_2 j_2}^{\gamma_2} Y^{\otimes m_2} \otimes Y^{*\otimes n_2}) : [Y=X, Y^\dagger=X^\dagger] \\ &= \sum_{m_1+1 \leq a \leq m_1+m_2, 1 \leq b \leq n_1} : tr_{m_1+m_2, n_1+n_2} (C_{ab} (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2}) \\ &\quad \times X^{\otimes a-1} \otimes 1 \otimes X^{m_1+m_2-a} \otimes X^{*\otimes b-1} \otimes 1 \otimes X^{*\otimes n_1+n_2-b}) : . \end{aligned} \quad (66)$$

The above two terms (63) and (66) can be combined to give

$$\sum_{\gamma, \vec{r}, i, j} \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, ij}^\gamma (D_{(1)} (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2})) : tr_{m_1+m_2-1, n_1+n_2-1} (Q_{\vec{r}, ij}^\gamma X^{\otimes m_1+m_2-1} \otimes X^{*\otimes n_1+n_2-1}) : , \quad (67)$$

where \vec{r} and γ run over irreducible representations of $S_{m_1+m_2-1} \times S_{n_1+n_2-1}$ and irreducible representations of $B_N(m_1+m_2-1, n_1+n_2-1)$ respectively. To write down the above form, we have introduced an operation $D_{(1)}$

$$D_{(1)} = \sum_{1 \leq a \leq m_1, n_1+1 \leq b \leq n_1+n_2} D_{ab} + \sum_{m_1+1 \leq a \leq m_1+m_2, 1 \leq b \leq n_1} D_{ab}, \quad (68)$$

where

$$\begin{aligned} D_{ab} &: (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2})_{j_1 \dots j_a \dots j_{m_1} j_{m_1+1} \dots j_{m_1+n_1}; l_1 \dots l_{m_2} l_{m_2+1} \dots l_b \dots l_{m_2+n_2}}^{i_1 \dots i_a \dots i_{m_1} i_{m_1+1} \dots i_{m_1+n_1}; k_1 \dots k_{m_2} k_{m_2+1} \dots k_b \dots k_{m_2+n_2}} \\ &\rightarrow \sum_{s, t} (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2})_{j_1 \dots t \dots j_{m_1} j_{m_1+1} \dots j_{m_1+n_1}; l_1 \dots l_{m_2} l_{m_2+1} \dots t \dots l_{m_2+n_2}}^{i_1 \dots s \dots i_{m_1} i_{m_1+1} \dots i_{m_1+n_1}; k_1 \dots k_{m_2} k_{m_2+1} \dots s \dots k_{m_2+n_2}}, \end{aligned} \quad (69)$$

for $1 \leq a \leq m_1, n_1 + 1 \leq b \leq n_1 + n_2$. Note that the operation D_{ab} is not a linear map on $V^{\otimes m_1+m_2} \otimes \bar{V}^{\otimes n_1+n_2}$, while C_{ab} is a linear map on it.

Thus the contribution of the term (67) in the correlation function (62) yields

$$\delta_{m, m_1+m_2-1} \delta_{n, n_1+n_2-1} m! n! Dim \gamma \chi_{\vec{r}, ij}^\gamma \left(D_{(1)} (Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2}) \right). \quad (70)$$

Defining

$$\begin{aligned} D_{(s)} &= \sum_{p=0}^s \frac{1}{(s-p)! p!} \left(\sum_{1 \leq a_1 \neq \dots \neq a_p \leq m_1} \sum_{n_1+1 \leq b_1 \neq \dots \neq b_p \leq n_1+n_2} D_{a_1 b_1} \dots D_{a_p b_p} \right) \\ &\quad \times \left(\sum_{m_1+1 \leq a_1 \neq \dots \neq a_{s-p} \leq m_1+m_2} \sum_{1 \leq b_1 \neq \dots \neq b_{s-p} \leq n_1} D_{a_1 b_1} \dots D_{a_{s-p} b_{s-p}} \right), \end{aligned} \quad (71)$$

the correlation function (62) can be computed as

$$\begin{aligned} & \langle : tr_{m_1, n_1}(Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} X^{\otimes m_1} \otimes X^{*\otimes n_1}) :: tr_{m_2, n_2}(Q_{\vec{r}_2, i_2 j_2}^{\gamma_2} X^{\otimes m_2} \otimes X^{*\otimes n_2}) :: tr_{m, n}(Q_{\vec{r}, ij}^{\gamma} X^{*\otimes m} \otimes X^{\otimes n}) : \rangle \\ &= m!n!Dim\gamma \sum_{0 \leq s \leq m_1+m_2-m} \delta_{m, m_1+m_2-s} \delta_{n, n_1+n_2-s} \chi_{\vec{r}, ij}^{\gamma} \left(D_{(s)}(Q_{\vec{r}_1, i_1 j_1}^{\gamma_1} \circ Q_{\vec{r}_2, i_2 j_2}^{\gamma_2}) \right). \end{aligned} \quad (72)$$

5 Correlation functions of the BPS operators

In this section, we shall turn our interest to a class of operators which are labelled by γ alone. Namely the operator we will consider is

$$O^{\gamma} = tr_{n_1+n_2, n_3}(P^{\gamma} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{T \otimes n_3}), \quad (73)$$

where $\gamma = (\gamma_+, \gamma_-) \vdash (n_1+n_2-k, n_3-k)$ ($0 \leq k \leq \min(n_1+n_2, n_3)$). The operator $P^{\gamma} = \sum_{\vec{r}, i} Q_{\vec{r}, ii}^{\gamma}$ is the projection operator for an irreducible representation γ . This class can attract a special interest because they are annihilated by the one-loop dilatation operator [19]. An essential part of computations can be captured by the two-matrix case, and hence we will study the case $n_2 = 0$ (and rename Z to Y , n_3 to n_2).

From the previous section, for three-point functions we have

$$\begin{aligned} & \langle O^{\gamma_1} O^{\gamma_2} O^{\gamma \dagger} \rangle \\ &= \langle tr_{m_1+m_2, n_1+n_2}((P^{\gamma_1} \circ P^{\gamma_2}) X^{\otimes m_1+m_2} \otimes Y^{T \otimes n_1+n_2}) tr_{m, n}(P^{\gamma} X^{\dagger \otimes m} \otimes Y^{*\otimes n}) \rangle \\ &= m!n! \delta_{m_1+m_2, m} \delta_{n_1+n_2, n} Dim\gamma \chi^{\gamma}(P^{\gamma_1} \circ P^{\gamma_2}) \\ &= m!n! \delta_{m_1+m_2, m} \delta_{n_1+n_2, n} Dim\gamma d_{\gamma_1} d_{\gamma_2} M_{\gamma_1, \gamma_2}^{\gamma}. \end{aligned} \quad (74)$$

d_{γ} is the dimension of γ considered as an irreducible representation of the Brauer algebra. In the last line above we have used the branching rule for $B_N(m, n) \rightarrow B_N(m_1, n_1) \times B_N(m_2, n_2)$ [30, 31]

$$V^{\gamma} \cong \bigoplus_{\gamma_1, \gamma_2} M_{\gamma_1, \gamma_2}^{\gamma} V^{\gamma_1} \otimes V^{\gamma_2}, \quad (75)$$

where $M_{\gamma_1, \gamma_2}^{\gamma}$ is the multiplicity that counts the number of times γ appears in the direct product of γ_1 and γ_2 :

$$M_{\gamma_1, \gamma_2}^{\gamma} = \sum_{\rho, \zeta, \theta, \kappa} \left(\sum_{\delta} g(\delta, \rho; \gamma_{1+}) g(\delta, \zeta; \gamma_{2-}) \right) \left(\sum_{\epsilon} g(\epsilon, \theta; \gamma_{1-}) g(\epsilon, \kappa; \gamma_{2+}) \right) g(\rho, \kappa; \gamma_+) g(\zeta, \theta; \gamma_-), \quad (76)$$

where $g(\delta, \rho; \gamma_{1+})$'s are the Littlewood-Richardson coefficients. If $\gamma_{1-} = 0$, $\gamma_{2-} = 0$, and $\gamma_- = 0$, we have

$$M_{(\gamma_{1+}, \emptyset), (\gamma_{2+}, \emptyset)}^{(\gamma_+, \emptyset)} = g(\gamma_{1+}, \gamma_{2+}; \gamma_+), \quad (77)$$

as expected.

Up to the normalisation factor, the three-point function is equivalent to the multiplicity associated with the restriction. Therefore information about the relevant physics is fully contained in the multiplicity and the normalisation. We shall try to extract physics by examining the multiplicity for several cases. In particular, one of our concerns is to know if there are relations among the integers k , k_1 , k_2 for non-zero correlators. In order to manifest the value of k we will often write representations like $\gamma = (\gamma_+, \gamma_-, k)$ or $\gamma(k)$.

Analysing the condition for the multiplicity to be non-zero, we obtain an inequality for k, k_1, k_2 :

$$k \geq k_1 + k_2. \quad (78)$$

We give the derivation in appendix E. The equality of (78) is $k = k_1 + k_2$, where the multiplicity takes the following form

$$M_{(\gamma_{1+}, \gamma_{1-}, k_1), (\gamma_{2+}, \gamma_{2-}, k_2)}^{(\gamma_+, \gamma_-, k_1 + k_2)} = g(\gamma_{1+}, \gamma_{2+}; \gamma_+) g(\gamma_{1-}, \gamma_{2-}; \gamma_-). \quad (79)$$

Here the γ_+ -sector and the γ_- -sector are completely decoupled. We call the form **factorised form**. In fact, $k = k_1 + k_2$ is a necessary and sufficient condition for the correlator to be factorised. For our convenience, we introduce a quantity

$$\Delta := k - (k_1 + k_2), \quad (80)$$

which cannot be negative due to (78). It measures the deviation of k from $k_1 + k_2$. Because $\Delta = 0$ is the case the correlator takes a factorised form, we expect that it can be a good index to measure how far the correlator is from the factorised form. We will see if Δ is really a good index in some concrete situations.

When γ takes the $k = 0$ representation, k_1 and k_2 are forced to be at $k_1 = 0$ and $k_2 = 0$ for the multiplicity to be non-zero. The multiplicity takes the factorised form

$$M_{\gamma_1(k=0), \gamma_2(k=0)}^{\gamma(k=0)} = g(\gamma_{1+}, \gamma_{2+}; \gamma_+) g(\gamma_{1-}, \gamma_{2-}; \gamma_-). \quad (81)$$

Here the number of boxes in each Young diagram is equal to the R-charge. The factorised form looks like we have two copies of the 1/2 BPS sector.

Consider the case where $\gamma_1 = (R, \emptyset)$, $\gamma_2 = (\emptyset, S)$ ($R \vdash m$, $S \vdash n$), i.e. the first operator is a Schur polynomial of X while the second operator is another Schur polynomial of Y . The multiplicity becomes

$$M_{R,S}^{(\gamma_+, \gamma_-, k)} = \sum_{\delta \vdash k} g(\delta, \gamma_+; R) g(\delta, \gamma_-; S). \quad (82)$$

In this case γ is allowed to take all possible values of k to have non-zero transitions.

We now try to give physical interpretations of the third operator. Suppose m and n are both $\mathcal{O}(N)$, and R and S are the symmetric representations or the anti-symmetric representations. These are just to have a concrete situation. The first operator and the second operator represent giant gravitons expanding in the S^5 or in the AdS_5 , but they have different angular directions, call J_1 and J_2 . For $k = 0$, we have non-zero transitions if and only if $\gamma_+ = R$ and $\gamma_- = S$. The third operator can be naturally considered to be a giant graviton with two angular momenta J_1 and J_2 or a composite of the two giant gravitons. If R and S are totally anti-symmetric, a group theoretic prediction (coming from the first one in (44)) is that the sum of the angular momenta should have cut-off at N . (This was called non-chiral stringy exclusion principle in [13].) The correlator can also be non-zero for $k \geq 1$. For $k \geq 1$, we come across a new situation in which the size of the Young diagrams does not represent the R-charge. When k is small, we give an interpretation that the third operator describes a system of a giant graviton whose size is determined by γ_+ and γ_- and a closed string excitation determined by the k . Increasing the value of k up to the possible

maximum value, the third operator is well described in terms of combined matrices like in (48). It would be a giant graviton that is different from the giant graviton at $k = 0$. In other words, a giant graviton whose size is determined by k would emerge when k is $\mathcal{O}(N)$. Because $\Delta = k$, increasing k is increasing Δ .

We shall next discuss cases in which the second operator is a Schur polynomial. The simplest case is the restriction $B_N(m, n) \rightarrow B_N(m-1, n)$, i.e. $\gamma_2 = ([1], 0)$. The operator labelled by γ_2 is just $\text{tr} X$, representing a KK graviton with a unit of angular momentum. The relevant multiplicity is as follows

$$M_{(\gamma_{1+}, \gamma_{1-}, k_1), ([1], \emptyset)}^{(\gamma_+, \gamma_-, k)} = \sum_{\epsilon, \kappa} g(\epsilon, \gamma_-; \gamma_{1-}) g(\epsilon, \kappa; [1]) g(\gamma_{1+}, \kappa; \gamma_+). \quad (83)$$

There are two cases to get non-zero multiplicities, which are $(\epsilon, \kappa) = (\emptyset, [1])$ and $([1], \emptyset)$. For the first case, we have

$$M_{(R, S, k_1), ([1], \emptyset)}^{(R_+, S, k)} = 1, \quad (84)$$

and zero otherwise. Here S is a Young diagram with $n - k$ boxes and R is a Young diagram with $m - 1 - k$ boxes. R_+ is a Young diagram obtained by adding a box to the R . We have $k = k_1$ ($0 \leq k \leq \min(m-1, n)$). For the second case, we have

$$M_{(R, S_+, k_1), ([1], \emptyset)}^{(R, S, k)} = 1, \quad (85)$$

and zero otherwise. R is a Young diagram with $m - k$ boxes and S is a Young diagram with $n - k$ boxes, and we have $k = k_1 + 1$ ($1 \leq k \leq \min(m, n)$). In this way, we find that the branching rule does not allow k and k_1 to take any possible values. They must be equal or be related by $k = k_1 + 1$. In terms of Δ , $\Delta = 0$ for the first case and $\Delta = 1$ for the second case.

Generalising the above case, consider a more general case with $\gamma_{2-} = 0$, i.e. $B_N(m, n) \rightarrow B_N(m_1, n) \times B_N(m_2, 0)$ ($m = m_1 + m_2$). We have

$$M_{\gamma_1, (T, \emptyset)}^\gamma = \sum_{\epsilon, \kappa} g(\epsilon, \gamma_-; \gamma_{1-}) g(\epsilon, \kappa; T) g(\gamma_{1+}, \kappa; \gamma_+), \quad (86)$$

where $T \vdash m_2$. For the multiplicity to be non-zero, k and k_1 cannot take any values, and the difference must satisfy the following

$$0 \leq \Delta = k - k_1 \leq m_2, \quad (87)$$

which we obtain by writing down consistency equations for the number of boxes of the Young diagrams in the LR coefficients. If we write $\gamma_1 = (\alpha, \beta, k_1)$, where $\alpha \vdash (m_1 - k_1)$, $\beta \vdash (n - k_1)$, non-zero multiplicities are obtained iff γ is given by

$$\gamma = (\alpha_{+s}, \beta_{-\Delta}, k = k_1 + \Delta) \quad (s = m_2 - \Delta). \quad (88)$$

Here α_{+s} is a Young diagram obtained by the tensor product of the α and a Young diagram with s boxes, and $\beta_{-\Delta}$ is a Young diagram that gives the β when the tensor product with a Young diagram with Δ boxes is considered.

When $\Delta = 0$, all boxes in T are added to α . This is the case of $(\epsilon, \kappa) = (\emptyset, T)$ in (86). The multiplicity is factorised

$$M_{(\gamma_{1+}, \gamma_{1-}, k_1), (T, \emptyset)}^{(\gamma_+, \gamma_-, k=k_1)} = g(\emptyset, \gamma_-; \gamma_{1-}) g(\gamma_{1+}, T; \gamma_+). \quad (89)$$

Shifting the value of Δ from zero, the correlator no longer takes a factorised form. Only $m_2 - \Delta$ boxes are added to α , and the remaining Δ boxes are added to $\beta_{-\Delta}$ to make β . When $\Delta = m_2$, all boxes in T are added to $\beta_{-\Delta}$. Thus Δ is a good index to know how far the correlator is from the factorised form.

We next think about cases where k takes the maximum possible value. Suppose $m = n$ just for simplicity. Because $\gamma = (\emptyset, \emptyset, k = m)$, the multiplicity takes the form

$$\begin{aligned} M_{\gamma_1, \gamma_2}^\gamma &= \sum_{\delta, \epsilon} g(\delta, \emptyset; \gamma_{1+}) g(\delta, \emptyset; \gamma_{2-}) g(\epsilon, \emptyset; \gamma_{1-}) g(\epsilon, \emptyset; \gamma_{2+}) \\ &= g(\gamma_{1+}, \emptyset; \gamma_{2-}) g(\gamma_{1-}, \emptyset; \gamma_{2+}). \end{aligned} \quad (90)$$

This is non-zero if we have $m_1 - k_1 = n_2 - k_2$. We know that $k_1 + k_2 = k = m_1 + m_2 = n_1 + n_2$ is a necessary and sufficient condition for the correlator to be factorised. Considering $0 \leq k_1 \leq \min(m_1, n_1)$ and $0 \leq k_2 \leq \min(m_2, n_2)$, it is satisfied iff we have $m_1 = n_1 = k_1$ and $m_2 = n_2 = k_2$. Hence the case $(\gamma_{1+}, \gamma_{1-}) = (\emptyset, \emptyset)$ and $(\gamma_{2+}, \gamma_{2-}) = (\emptyset, \emptyset)$ is the only case the correlator is factorised.

When $k_2 = m_2 = n_2$ with arbitrary γ_1, γ , we have the non-zero multiplicities given by

$$M_{(\gamma_{1+}, \gamma_{1-}, k_1), (\emptyset, \emptyset, k_2)}^{(\gamma_{1+}, \gamma_{1-}, k)} = 1 \quad (91)$$

with $m_1 - k_1 = m - k$. The correlator is already taking a factorised form. It is consistent because we always have

$$k = k_1 + k_2, \quad (92)$$

which comes from R-charge conservation $m_1 + m_2 = m$. It is interesting that (92) is a reflection of R-charge conservation.

Finally we present a more general correlation function

$$\begin{aligned} &\langle O^{\gamma_1} \dots O^{\gamma_s} O^{\gamma_{s+1}^\dagger} \dots O^{\gamma_{s+t}^\dagger} \rangle \\ &= m!n! \sum_{\gamma, \vec{r}, ij} \text{Dim} \gamma \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, ij}^\gamma (P^{\gamma_1} \circ \dots \circ P^{\gamma_s}) \chi_{\vec{r}, ji}^\gamma (P^{\gamma_{s+1}} \circ \dots \circ P^{\gamma_{s+t}}). \end{aligned} \quad (93)$$

Here γ is an irreducible representation of $B_N(m, n)$, and \vec{r} is an irreducible representation of $\mathbb{C}[S_m \times S_n]$. We need $m = m_1 + \dots + m_s = m_{s+1} + \dots + m_{s+t}$ and $n = n_1 + \dots + n_s = n_{s+1} + \dots + n_{s+t}$ to obtain a non-zero result. When $t = 1$, we get

$$\begin{aligned} \langle O^{\gamma_1} O^{\gamma_2} \dots O^{\gamma_s} O^{\gamma^\dagger} \rangle &= m!n! \text{Dim} \gamma \chi^\gamma (P^{\gamma_1} \circ P^{\gamma_2} \circ \dots \circ P^{\gamma_s}) \\ &= m!n! \text{Dim} \gamma d_{\gamma_1} d_{\gamma_2} \dots d_{\gamma_s} M_{\gamma_1, \gamma_2, \dots, \gamma_s}^\gamma, \end{aligned} \quad (94)$$

where

$$M_{\gamma_1, \gamma_2, \dots, \gamma_s}^\gamma := \sum_{r, q, \dots, r} M_{\gamma_1 p}^\gamma M_{\gamma_2 q}^p \dots M_{\gamma_{s-1} \gamma_s}^r. \quad (95)$$

It is non-zero if the following inequality is satisfied

$$k \geq k_1 + k_2 + \dots + k_s, \quad (96)$$

which is an extension of (78). If the equality is the case, $k = k_1 + k_2 + \dots + k_s$, the correlator takes the factorised form

$$\langle O^{\gamma_1} O^{\gamma_2} \dots O^{\gamma_s} O^{\gamma^\dagger} \rangle = m!n! \text{Dim} \gamma \, g(\gamma_{1+}, \gamma_{2+}, \dots; \gamma_+) g(\gamma_{1-}, \gamma_{2-}, \dots; \gamma_-). \quad (97)$$

It is interesting that the matrix integral is decomposed into the two pieces.

6 Discussions

In this paper, we have studied correlation functions of local gauge invariant operators in $\mathcal{N} = 4$ SYM at zero coupling by starting from the review of the symmetries of the free theory. In particular we have studied a basis that uses the Brauer algebra in more detail for the $so(6)$ scalar sector. Our construction of bases followed the guideline that comes from the structure of commuting conserved charges.

Inclusion of other fields is similarly managed. The Hamiltonian of the free theory on $R \times S^3$ can be written as a set of infinitely many harmonic oscillators [32, 33, 34]. Suppose we are interested in an s -oscillator system. A representation basis can be labelled by two kinds of Young diagrams. One is a set of Young diagrams $(\alpha_1, \dots, \alpha_s)$ corresponding to an irreducible representation of $S_{n_1} \times \dots \times S_{n_s}$. This is related to the fact that the system has s towers of conserved charges, corresponding to the set S for the case of the $su(3)$ sector. The other is a Young diagram R or a pair of Young diagrams (γ_+, γ_-) corresponding to an irreducible representation of $S_{n_1 + \dots + n_s}$ or an irreducible representation of the Brauer algebra $B_N(n_+, n_-)$, where $n_+ + n_- = n_1 + \dots + n_s$. We may choose whichever basis we like in order to have an orthogonal basis, but it would be interesting to find that both have a common label corresponding to an irreducible representation of $S_{n_1} \times \dots \times S_{n_s}$. This might suggest that we can furnish a universal physical meaning to the eigenvalues $C_p(\alpha_i)$ of the conserved charges in the context of a string theory or a higher-spin field theory on AdS.

In the latter part of this paper, we have computed exact correlation functions of the Brauer basis at zero coupling. Multi-point correlation functions for a class of 1/4 BPS operators were written down by a branching rule of the Brauer algebra - see (74) and (94). We have found that the multi-point correlation functions take a factorised form in which the γ_+ -sector and the γ_- -sector are completely decoupled if k 's satisfy a relation - the equality in (78). This might suggest a fascinating possibility that general correlation functions for the Brauer operators are well classified in terms of the integers. The meaning of k was studied in the context of the correspondence between 1/4 BPS bubbling geometries and the 1/4 BPS Brauer operators in [29]. It was naturally interpreted as the mixing between the two angular directions. It will be an interesting future direction to reinforce this interpretation from a study of correlation functions that contain large operators using the techniques developed in [35, 36, 37].

We could see some similarities between the Brauer basis and the restricted Schur basis. They have a common set of conserved charges (which we denoted by S), and the Brauer basis contains the restricted Schur basis as a subset in the sense of (46) and (48). We are wondering if the Brauer operator admits a quantum non-planar integrability similar to the recent observations in [40, 41, 42, 43, 44, 45, 46, 47, 48]. See also [38, 39] for the construction of BPS operators at weak coupling. It will be interesting to develop group theoretic methods for extracting universal features of non-planar theories.

Acknowledgements

I would like to thank Robert de Mello Koch and Sanjaye Ramgoolam for reading the draft and making helpful comments.

A Representation theory of the Brauer algebra

In this appendix, we briefly summarise the representation theory of the Brauer algebra $B_N(m, n)$. See also the paper [13], and references therein.

The Brauer algebra, which we denote by $B_N(m, n)$, naturally appears in the decomposition of the following tensor product representation

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_N(m, n)}, \quad (\text{A.1})$$

where V is the fundamental representation of $U(N)$. This equation comes from the fact that the $U(N)$ action commutes with the action of the Brauer algebra on the tensor product. In the equation, γ is an irreducible representation of the Brauer algebra and $U(N)$. It is given by a pair of two Young diagrams which have $m - k$ boxes and $n - k$ boxes, and k is an integer satisfying $0 \leq k \leq \min(m, n)$. Taking this into account, the sum of γ can be re-grouped to be

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_k \left(\bigoplus_{\gamma_+ \vdash (m-k), \gamma_- \vdash (n-k)} V_{(\gamma_+, \gamma_-)}^{U(N)} \otimes V_{(\gamma_+, \gamma_-)}^{B_N(m, n)} \right). \quad (\text{A.2})$$

The Brauer algebra contains the group algebra of the symmetric group $S_m \times S_n$ as a subalgebra:

$$V_{\gamma}^{B_N(m, n)} = \bigoplus_{\alpha \vdash m, \beta \vdash n} V_{\gamma \rightarrow (\alpha, \beta)} \otimes V_{\alpha}^{\mathbb{C}[S_m]} \otimes V_{\beta}^{\mathbb{C}[S_n]}. \quad (\text{A.3})$$

The vector space $V_{\gamma \rightarrow (\alpha, \beta)}$ accounts for multiplicities in the restriction. The dimension of the space, $M_{(\alpha, \beta)}^{\gamma} := \text{Dim} V_{\gamma \rightarrow (\alpha, \beta)}$, counts the number of times the irreducible representation (α, β) appears in the irreducible representation γ by the formula

$$M_{(\alpha, \beta)}^{\gamma} = \sum_{\delta \vdash k} g(\delta, \gamma_+; \alpha) g(\delta, \gamma_-; \beta). \quad (\text{A.4})$$

When $k = 0$, we have

$$V_{(\alpha, \beta)}^{B_N(m, n)} = V_{\alpha}^{\mathbb{C}[S_m]} \otimes V_{\beta}^{\mathbb{C}[S_n]}. \quad (\text{A.5})$$

B The conserved charges

In this appendix, we present some sets of commuting conserved charges for the sector with excitations by B_a^{\dagger} ($a = 1, 2, 3$). The extension to more oscillators is straightforward. We will use (11) as building blocks. It is trivial to find that the following operators commute each other

$$\text{tr}((G_{L1}^B)^p), \quad \text{tr}((G_{L2}^B)^p), \quad \text{tr}((G_{L3}^B)^p), \quad \text{tr}((G_{R1}^B)^p), \quad \text{tr}((G_{R2}^B)^p), \quad \text{tr}((G_{R3}^B)^p). \quad (\text{B.1})$$

We call this set S . We find that it is helpful to use the following formula to show several things

$$[\text{tr}(G^p), (G^q)_{ij}] = 0, \quad (\text{B.2})$$

where G is an operator satisfying the $u(N)$ commutation relation. We choose $G = G_{L1}^B, G_{L2}^B + G_{L3}^B$, and so on. From the formula, we find that any charges built from the building blocks commute with the charges in S .

A set of commuting higher charges is given by

$$\begin{aligned} S, \quad & \text{tr}((G_{L1} + G_{L2} + G_{L3})^p), \quad \text{tr}((G_{L1} + G_{L2})^q), \\ & \text{tr}((G_{R1} + G_{R2} + G_{R3})^p), \quad \text{tr}((G_{R1} + G_{R2})^q). \end{aligned} \quad (\text{B.3})$$

This is related to the restricted Schur basis in the sense that these charges have good actions on the restricted Schur basis as we explicitly show in the next appendix. We can verify that all of these charges commute each other with the help of (B.2). In stead of $\text{tr}((G_{L1} + G_{L2})^q)$, we could put $\text{tr}((G_{L1} + G_{L3})^q)$ but could not put them together.

Taking into account the fact that $G_{L1} + G_{L2}$ is a $u(N)$ generator, we can find that

$$[(G_{L1} + G_{L2})_{ij}, \text{tr}(H)] = 0, \quad (\text{B.4})$$

where H is a polynomial built from G_{L1} and G_{L2} . With the help of this equation, we find the operator $\text{tr}(H)$ commutes with all operators in (B.3). But in general the trace of a polynomial of G_{L1} and G_{L2} does not commutes with the trace of another polynomial of G_{L1} and G_{L2} . Similarly, the trace of a polynomial built from $G_{L1} + G_{L2}$ and G_{L3} commutes with all operators in (B.3) and $\text{tr}(H)$ thanks to (B.2), (B.4), and the following

$$[(G_{L1} + G_{L2} + G_{L3})_{ij}, \text{tr}(I)] = 0, \quad (\text{B.5})$$

where I is a polynomial built from $G_{L1} + G_{L2}$ and G_{L3} . For example, we can choose $H = (G_{L1})^2(G_{L2})$, $I = (G_{L1} + G_{L2})^2(G_{L3})$, and we also have $H' = (G_{R1})^2(G_{R2})$, $I' = (G_{R1} + G_{R2})^2(G_{R3})$. These charges would be relevant for the multiplicity indices [17].

Another set of commuting higher charges is

$$\begin{aligned} S, \quad & \text{tr}((G_{L1} + G_{L2} + G_{R3})^p), \quad \text{tr}((G_{L1} + G_{L2})^q), \\ & \text{tr}((G_{R1} + G_{R2} + G_{L3})^p), \quad \text{tr}((G_{R1} + G_{R2})^q). \end{aligned} \quad (\text{B.6})$$

This is closely related to the Brauer basis $B_N(n_1 + n_2, n_3)$. We may put $\text{tr}((G_{L1} + G_{R3})^q)$ and $\text{tr}((G_{R1} + G_{L3})^q)$ instead of $\text{tr}((G_{L1} + G_{L2})^q)$ and $\text{tr}((G_{R1} + G_{R2})^q)$ in (B.6), but we cannot put all of these at the same time. We are allowed to include more charges, for example, $\text{tr}((G_{L1})^2(G_{L2}))$, $\text{tr}((G_{L1} + G_{L2})^2(G_{R3}))$, $\text{tr}((G_{R1})^2(G_{R2}))$, and $\text{tr}((G_{R1} + G_{R2})^2(G_{L3}))$.

By a similar construction to the previous two cases, we can form a set of commuting charges that includes

$$S, \quad \text{tr}((G_{L1} + G_{L2} + G_{R1})^p), \quad \text{tr}((G_{R2} + G_{R3} + G_{L3})^p). \quad (\text{B.7})$$

These charges are simultaneously diagonalised by operators of the form $(P_{r,ij}^\gamma)^{MLN} X_J^I Y_L^K Z_N^{TM}$ in terms of the Brauer algebra. But considering carefully, we realise that such operators do not form a complete set of local gauge invariant operators⁶, in general, though free two-point functions are diagonalised.

⁶ The following is an exception. Consider $m = n = 1$ in the two-matrix system. We have two gauge invariant operators, $\text{tr}XY$ and $\text{tr}X\text{tr}Y$. From these operators, we can find some combinations which have diagonal free two-point functions: (1) $(\text{tr}X\text{tr}Y + \text{tr}XY, \text{tr}X\text{tr}Y - \text{tr}XY)$, (2) $(\text{tr}X\text{tr}Y - (1/N)\text{tr}XY, \text{tr}XY)$, (3) $(\text{tr}XY - \frac{1}{N}\text{tr}X\text{tr}Y, \text{tr}X\text{tr}Y)$. The first one is the restricted Schur basis, while the second one is the Brauer basis. The last one is the new basis related to the charges (B.7).

C Action of the conserved charges

In this section, we display eigenvalues of the charges given in appendix B. For more detail see [17].

The Schur polynomial basis relevant to the 1/2 BPS sector is given by

$$|R\rangle = tr_n(p_R B_1^{\dagger \otimes n})|0\rangle = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma^{-1}) tr_n(\sigma B_1^{\dagger \otimes n})|0\rangle, \quad (C.1)$$

where R is a Young diagram with n boxes ($c_1(R) \leq N$). This basis has the following actions

$$\begin{aligned} tr((B_1^{\dagger} B_1)^p)|R\rangle &= C_p(R)|R\rangle, \\ tr((B_1 B_1^{\dagger})^p)|R\rangle &= C_p(R)|R\rangle. \end{aligned} \quad (C.2)$$

$C_p(R)$ is the p -th Casimir.

For the restricted Schur basis

$$|R, \vec{r}, ij\rangle = tr_{n_1+n_2+n_3}(P_{\vec{r}, ij}^R (B_1^{\dagger})^{\otimes n_1} \otimes (B_2^{\dagger})^{\otimes n_2} \otimes (B_3^{\dagger})^{\otimes n_3})|0\rangle, \quad (C.3)$$

we have the following actions of the commuting conserved charges

$$\begin{aligned} tr((G_{L1})^p)|R, \vec{r}, ij\rangle &= C_p(r_1)|R, \vec{r}, ij\rangle, \\ tr((G_{L2})^p)|R, \vec{r}, ij\rangle &= C_p(r_2)|R, \vec{r}, ij\rangle, \\ tr((G_{L3})^p)|R, \vec{r}, ij\rangle &= C_p(r_3)|R, \vec{r}, ij\rangle, \\ tr((G_{L1} + G_{L2} + G_{L3})^p)|R, \vec{r}, ij\rangle &= C_p(R)|R, \vec{r}, ij\rangle, \\ tr((G_{L1} + G_{L2})^p)|R, \vec{r}, ij\rangle &= \sum_{\alpha \vdash (n_1+n_2)} C_p(\alpha) \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, jl}^R(P^{\alpha} \circ 1)|R, \vec{r}, il\rangle, \\ tr((G_{R1} + G_{R2})^p)|R, \vec{r}, ij\rangle &= \sum_{\alpha \vdash (n_1+n_2)} C_p(\alpha) \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, ki}^R(P^{\alpha} \circ 1)|R, \vec{r}, kj\rangle. \end{aligned} \quad (C.4)$$

The last two are derived by inserting $1 = \sum_{\alpha} P^{\alpha}$ and using an equation similar to (43).

As was mentioned around (B.4) (B.5), we have more charges. Let us choose $tr((G_{L1} + G_{L2})^2 G_{L3})$ and $tr(G_{L1}^2 G_{L2})$ for instance. The exact eigenvalues have not been clear, but it was guessed in [17] that those will be measuring the multiplicity indices. Our guess is that the first one is related to the restriction $S_{n_1+n_2+n_3} \rightarrow S_{n_1+n_2} \times S_{n_3}$ and the second one is related to the restriction $S_{n_1+n_2} \rightarrow S_{n_1} \times S_{n_2}$.

Likewise for the Brauer basis

$$|\gamma, \vec{r}, ij\rangle = tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^{\gamma} (B_1^{\dagger})^{\otimes n_1} \otimes (B_2^{\dagger})^{\otimes n_2} \otimes (B_3^{\dagger})^{T \otimes n_3})|0\rangle, \quad (C.5)$$

we have

$$\begin{aligned}
tr((G_{L1})^p)|\gamma, \vec{r}, ij\rangle &= C_p(r_1)|\gamma, \vec{r}, ij\rangle, \\
tr((G_{L2})^p)|\gamma, \vec{r}, ij\rangle &= C_p(r_2)|\gamma, \vec{r}, ij\rangle, \\
tr((G_{L3})^p)|\gamma, \vec{r}, ij\rangle &= C_p(r_3)|\gamma, \vec{r}, ij\rangle, \\
tr((G_{L1} + G_{L2} + G_{R3})^p)|\gamma, \vec{r}, ij\rangle &= C_p(\gamma)|\gamma, \vec{r}, ij\rangle, \\
tr((G_{L1} + G_{L2})^p)|\gamma, \vec{r}, ij\rangle &= \sum_{\alpha \vdash (n_1+n_2)} C_p(\alpha) \frac{1}{d_{\vec{r}}} \chi_{\vec{r},jl}^\gamma(P^\alpha \circ 1)|\gamma, \vec{r}, il\rangle, \\
tr((G_{R1} + G_{R2})^p)|\gamma, \vec{r}, ij\rangle &= \sum_{\alpha \vdash (n_1+n_2)} C_p(\alpha) \frac{1}{d_{\vec{r}}} \chi_{\vec{r},ki}^\gamma(P^\alpha \circ 1)|\gamma, \vec{r}, kj\rangle, \\
tr((G_{L1} + G_{R3})^p)|\gamma, \vec{r}, ij\rangle &= \sum_{\gamma_1} C_p(\gamma_1) \frac{1}{d_{\vec{r}}} \chi_{\vec{r},jl}^\gamma(P^{\gamma_1})|\gamma, \vec{r}, il\rangle, \\
tr((G_{R1} + G_{L3})^p)|\gamma, \vec{r}, ij\rangle &= \sum_{\gamma_1} C_p(\gamma_1) \frac{1}{d_{\vec{r}}} \chi_{\vec{r},ki}^\gamma(P^{\gamma_1})|\gamma, \vec{r}, kj\rangle.
\end{aligned}$$

In the last two equations, γ_1 runs over irreducible representations of $B_N(n_1, n_3)$. The 5th and 6th equations take the following forms at $k = 0$

$$\begin{aligned}
tr((G_{L1} + G_{L2})^p)|(\gamma_+, \gamma_-, k=0), \vec{r}, ij\rangle &= C_p(\gamma_+)|(\gamma_+, \gamma_-, k=0), \vec{r}, ij\rangle, \\
tr((G_{R1} + G_{R2})^p)|(\gamma_+, \gamma_-, k=0), \vec{r}, ij\rangle &= C_p(\gamma_+)|(\gamma_+, \gamma_-, k=0), \vec{r}, ij\rangle.
\end{aligned} \tag{C.6}$$

D Two special sectors of the Brauer basis

In this appendix, we will show explicit forms of the operator in the two sectors where the integer k takes the minimum possible value and the maximum possible value.

D.1 $k = 0$

When k takes zero, we have $\gamma_- = r_3$ and the multiplicity indices on the $k = 0$ operators run over from 1 to

$$M_{\vec{r}}^{\gamma(k=0)} = g(r_1, r_2; \gamma_+). \tag{D.1}$$

In this sector, some special properties are available to rewrite the form of the operator $Q_{\vec{r},ij}^{\gamma(k=0)}$ [13]:

$$\begin{aligned}
Q_{\vec{r},ij}^{\gamma(k=0)} &= Dim\gamma \sum_b \chi_{\vec{r},ij}^\gamma(b) b^* \\
&= Dim\gamma \sum_{b \in \mathbb{C}[S_{n_1+n_2} \times S_{n_3}]} \chi_{\vec{r},ij}^\gamma(b) b^* \\
&= Dim\gamma \sum_{\sigma \in S_{n_1+n_2}} \sum_{\tau \in S_{n_3}} \chi_{(r_1, r_2), ij}^{\gamma_+}(\sigma) \chi_{r_3}(\tau) 1^*(\sigma \circ \tau)^{-1} \\
&= Dim\gamma \frac{(n_1 + n_2)!}{d_{\gamma_+}} \frac{n_3!}{d_{r_3}} 1^* P_{(r_1, r_2), ij}^{\gamma_+} p_{r_3},
\end{aligned} \tag{D.2}$$

where

$$P_{(r_1, r_2), ij}^{\gamma+} = \frac{d_{r_+}}{(n_1 + n_2)!} \sum_{\sigma \in S_{n_1+n_2}} \chi_{(r_1, r_2), ij}^{\gamma+}(\sigma) \sigma^{-1}. \quad (\text{D.3})$$

1^* is a specific element expressed by a linear combination of elements in the Brauer algebra [13]. Using a formula of 1^* , we have

$$\begin{aligned} & tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^{\gamma(k=0)} X \otimes Y \otimes Z^T) \\ = & Dim\gamma \frac{(n_1 + n_2)!}{d_{\gamma_-}} \frac{n_3!}{d_{r_3}} \frac{1}{N^n} tr_n(\Omega_n^{-1} P_{(r_1, r_2), ij}^{\gamma+} p_{r_3} X \otimes Y \otimes Z), \end{aligned} \quad (\text{D.4})$$

where $n \equiv n_1 + n_2 + n_3$. Note that Z 's are not transposed. It is an exercise to express it as a linear combination of restricted Schur polynomials. Making use of

$$\begin{aligned} \Omega_n^{-1} P_{(r_1, r_2), ij}^{\gamma+} p_{r_3} &= \sum_{S, \vec{s}, kl} \frac{1}{d_s} \chi_{s, kl}^S \left(\Omega_n^{-1} P_{(r_1, r_2), ij}^{\gamma+} p_{r_3} \right) P_{s, kl}^S \\ &= \sum_{S, kl} \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, kl}^S \left(P_{(r_1, r_2), ij}^{\gamma+} \right) \frac{N^n}{n!} \frac{d_s}{DimS} P_{\vec{r}, kl}^S, \end{aligned} \quad (\text{D.5})$$

where the second line is obtained by using

$$\Omega_n^{-1} = \frac{N^n}{n!} \sum_{S \vdash n} \frac{d_S}{DimS} P^S, \quad (\text{D.6})$$

we obtain

$$\begin{aligned} & tr_{n_1+n_2, n_3}(Q_{\vec{r}, ij}^{\gamma(k=0)} X \otimes Y \otimes Z^T) \\ = & Dim\gamma \frac{(n_1 + n_2)!}{d_{r_+}} \frac{n_3!}{d_{r_3}} \frac{1}{N^n} tr_n(\Omega_n^{-1} P_{(r_1, r_2), ij}^{\gamma+} p_{r_3} X \otimes Y \otimes Z) \\ = & Dim\gamma \frac{(n_1 + n_2)! n_3!}{n!} \frac{1}{d_{r_+} d_{r_3}} \sum_{S, kl} \frac{1}{d_{\vec{r}}} \chi_{\vec{r}, kl}^S \left(P_{(r_1, r_2), ij}^{\gamma+} \right) \frac{d_s}{DimS} tr_n(P_{\vec{r}, kl}^S X \otimes Y \otimes Z). \end{aligned} \quad (\text{D.7})$$

D.2 $k = n_1 + n_2 = n_3$

We consider the case $n_1 + n_2 = n_3$ for simplicity. When the integer k takes the maximum possible value $k = n_1 + n_2 = n_3$, $\gamma = (\emptyset, \emptyset)$. The multiplicity is given by

$$M_{\vec{r}}^{\gamma} = g(r_1, r_2; r_3). \quad (\text{D.8})$$

First introduce the contraction operator [22]

$$C_{(k)} = \sum_{\sigma \in S_k} \sigma C_{1\bar{1}} \cdots C_{k\bar{k}} \sigma^{-1}, \quad (\text{D.9})$$

which satisfies

$$C_{(k)}^2 = N^k \Omega_k C_{(k)}. \quad (\text{D.10})$$

In terms of this, we can construct the operator

$$Q_{\vec{r},ij}^{\gamma(k=n_1+n_2=n_3)} = \frac{d_{r_3}}{Dimr_3n_3!} (P_{(r_1,r_2),ij}^{r_3} \otimes 1) C_{(k)}. \quad (D.11)$$

We will check it satisfies the following required relations

$$\begin{aligned} Q_{\vec{r},ij} Q_{\vec{s},kl} &= \delta_{\vec{r}\vec{s}} \delta_{jk} Q_{\vec{r},il}, \\ tr_{n_1+n_2,n_3} (Q_{\vec{r},ij}^{\gamma(k=n_1+n_2=n_3)}) &= d_{\vec{r}} \delta_{ij}. \end{aligned} \quad (D.12)$$

The proof of the first equation is as follows:

$$\begin{aligned} & (P_{(r_1,r_2),ij}^{r_3} \otimes 1) C_{(k)} (P_{(s_1,s_2),kl}^{s_3} \otimes 1) C_{(k)} \\ &= \delta_{\vec{r}\vec{s}} \delta_{jk} (P_{(r_1,r_2),il}^{r_3} \otimes 1) C_{(k)}^2 \\ &= \delta_{\vec{r}\vec{s}} \delta_{jk} N^k \Omega_k (P_{(r_1,r_2),ij}^{r_3} \otimes 1) C_{(k)} \\ &= \delta_{\vec{r}\vec{s}} \delta_{jk} \frac{Dimr_3n_3!}{d_{r_3}} (P_{(r_1,r_2),ij}^{r_3} \otimes 1) C_{(k)}. \end{aligned} \quad (D.13)$$

We can also verify the second equation as

$$\begin{aligned} & tr_{n_1+n_2,n_3} (Q_{\vec{r},ij}^{\gamma(k=n_1+n_2=n_3)}) \\ &= \frac{d_{r_3}}{Dimr_3n_3!} \sum_{\sigma \in S_k} tr_{n_1+n_2} (\sigma P_{(r_1,r_2),ij}^{r_3} \sigma^{-1}) \\ &= \frac{d_{r_3}}{Dimr_3} tr_{n_1+n_2} (P_{(r_1,r_2),ij}^{r_3}) \\ &= \frac{d_{r_3}}{Dimr_3} Dimr_3 d_{r_1} d_{r_2} \delta_{ij} \\ &= d_{r_1} d_{r_2} d_{r_3} \delta_{ij}. \end{aligned} \quad (D.14)$$

The operator looks like

$$\begin{aligned} & tr_{n_1+n_2,n_3} (Q_{(r_1,r_2,r_3),ij}^{\gamma(k=n_1+n_2=n_3)} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{T \otimes n_3}) \\ &= \frac{d_{r_3}}{Dimr_3n_3!} tr_{n_1+n_2,n_3} ((P_{(r_1,r_2),ij}^{r_3} \otimes 1) C_{(k)} X^{\otimes n_1} \otimes Y^{\otimes n_2} \otimes Z^{T \otimes n_3}) \\ &= \frac{d_{r_3}}{Dimr_3} tr_{n_1+n_2} (P_{(r_1,r_2),ij}^{r_3} (ZX)^{\otimes n_1} \otimes (ZY)^{\otimes n_2}). \end{aligned} \quad (D.15)$$

This is nothing but a restricted Schur polynomial built from the two matrices ZX and ZY .

E The branching rule of the Brauer algebra

The branching rule for $B_N(m_1, n_1) \times B_N(m_2, n_2) \subseteq B_N(m, n)$ is given by

$$V_\gamma = \bigoplus_{\gamma_1, \gamma_2} M_{\gamma_1, \gamma_2}^\gamma V_{\gamma_1} \otimes V_{\gamma_2}, \quad (E.1)$$

where γ_1 , γ_2 , and γ are irreducible representations of $B_N(m_1, n_1)$, $B_N(m_2, n_2)$ and $B_N(m, n)$, respectively. The multiplicity $M_{\gamma_1, \gamma_2}^\gamma$ is expressed in terms of the Littlewood-Richardson coefficient

as [30, 31]

$$M_{\gamma_1, \gamma_2}^\gamma = \sum_{\rho, \zeta, \theta, \kappa} \left(\sum_{\delta} g(\delta, \rho; \gamma_{1+}) g(\delta, \zeta; \gamma_{2-}) \right) \left(\sum_{\epsilon} g(\epsilon, \theta; \gamma_{1-}) g(\epsilon, \kappa; \gamma_{2+}) \right) g(\rho, \kappa; \gamma_+) g(\zeta, \theta; \gamma_-). \quad (\text{E.2})$$

Denote the number of boxes contained in a Young diagram α by $n(\alpha)$. From consistency conditions for the Littlewood-Richardson coefficients, we need the following conditions

$$\begin{aligned} n(\delta) + n(\rho) &= m_1 - k_1, \\ n(\delta) + n(\zeta) &= n_2 - k_2, \\ n(\epsilon) + n(\theta) &= n_1 - k_1, \\ n(\epsilon) + n(\kappa) &= m_2 - k_2, \\ n(\rho) + n(\kappa) &= m - k, \\ n(\zeta) + n(\theta) &= n - k. \end{aligned} \quad (\text{E.3})$$

We also have

$$m_1 + m_2 = m, \quad n_1 + n_2 = n. \quad (\text{E.4})$$

From these conditions, we have

$$n(\delta) + n(\epsilon) = k - (k_1 + k_2). \quad (\text{E.5})$$

For this to be satisfied for any Young diagrams δ, ϵ , one should have

$$k - (k_1 + k_2) \geq 0. \quad (\text{E.6})$$

The equality is if and only if $n(\delta) = 0$ and $n(\epsilon) = 0$, which is the case $m_1 - k_1 + m_2 - k_2 = m - k$ and $n_1 - k_1 + n_2 - k_2 = n - k$. The multiplicity becomes for this case

$$M_{\gamma_1, \gamma_2}^{\gamma(k=k_1+k_2)} = g(\gamma_{1+}, \gamma_{2+}; \gamma_+) g(\gamma_{1-}, \gamma_{2-}; \gamma_-). \quad (\text{E.7})$$

The γ_+ -sector and the γ_- -sector are completely decoupled. We call this form factorised form.

References

- [1] N. Beisert et al., “Review of AdS/CFT Integrability: An Overview,” *Lett. Math. Phys.* **99** (2012) 3 [arXiv:1012.3982 [hep-th]].
- [2] C. Kristjansen, “Review of AdS/CFT Integrability, Chapter IV.1: Aspects of Non-Planarity,” *Lett. Math. Phys.* **99** (2012) 349 [arXiv:1012.3997 [hep-th]].
- [3] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from Anti-de Sitter space,” *JHEP* **0006** (2000) 008 [hep-th/0003075].
- [4] M. T. Grisaru, R. C. Myers and O. Tafjord, “SUSY and goliath,” *JHEP* **0008** (2000) 040 [hep-th/0008015].

- [5] A. Hashimoto, S. Hirano and N. Itzhaki, “Large branes in AdS and their field theory dual,” JHEP **0008** (2000) 051 [hep-th/0008016].
- [6] H. Lin, O. Lunin and J. M. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP **0410** (2004) 025 [hep-th/0409174].
- [7] P. Haggi-Mani and B. Sundborg, “Free large N supersymmetric Yang-Mills theory as a string theory,” JHEP **0004** (2000) 031 [hep-th/0002189].
- [8] B. Sundborg, “Stringy gravity, interacting tensionless strings and massless higher spins,” Nucl. Phys. Proc. Suppl. **102** (2001) 113 [hep-th/0103247].
- [9] E. Sezgin and P. Sundell, “Massless higher spins and holography,” Nucl. Phys. B **644** (2002) 303 [Erratum-ibid. B **660** (2003) 403] [hep-th/0205131].
- [10] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, “Higher spin symmetry and N=4 SYM,” JHEP **0407** (2004) 058 [hep-th/0405057].
- [11] S. Corley, A. Jevicki and S. Ramgoolam, “Exact correlators of giant gravitons from dual N=4 SYM theory,” Adv. Theor. Math. Phys. **5** (2002) 809 [hep-th/0111222].
- [12] S. Corley and S. Ramgoolam, “Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory,” Nucl. Phys. B **641** (2002) 131 [hep-th/0205221].
- [13] Y. Kimura and S. Ramgoolam, “Branes, anti-branes and brauer algebras in gauge-gravity duality,” JHEP **0711** (2007) 078 [arXiv:0709.2158 [hep-th]].
- [14] T. W. Brown, P. J. Heslop and S. Ramgoolam, “Diagonal multi-matrix correlators and BPS operators in N=4 SYM,” JHEP **0802** (2008) 030 [arXiv:0711.0176 [hep-th]].
- [15] R. Bhattacharyya, S. Collins and R. d. M. Koch, “Exact Multi-Matrix Correlators,” JHEP **0803** (2008) 044 [arXiv:0801.2061 [hep-th]].
- [16] T. W. Brown, P. J. Heslop and S. Ramgoolam, “Diagonal free field matrix correlators, global symmetries and giant gravitons,” JHEP **0904** (2009) 089 [arXiv:0806.1911 [hep-th]].
- [17] Y. Kimura and S. Ramgoolam, “Enhanced symmetries of gauge theory and resolving the spectrum of local operators,” Phys. Rev. D **78** (2008) 126003 [arXiv:0807.3696 [hep-th]].
- [18] Y. Kimura, “Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra,” JHEP **0912** (2009) 044 [arXiv:0910.2170 [hep-th]].
- [19] Y. Kimura, “Quarter BPS classified by Brauer algebra,” JHEP **1005** (2010) 103 [arXiv:1002.2424 [hep-th]].
- [20] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “A New double scaling limit of N=4 superYang-Mills theory and PP wave strings,” Nucl. Phys. B **643** (2002) 3 [hep-th/0205033].
- [21] D. Berenstein, “A Toy model for the AdS / CFT correspondence,” JHEP **0407** (2004) 018 [hep-th/0403110].

- [22] Y. Kimura, S. Ramgoolam and D. Turton, “Free particles from Brauer algebras in complex matrix models,” JHEP **1005** (2010) 052 [arXiv:0911.4408 [hep-th]].
- [23] A. P. Polychronakos, “Physics and Mathematics of Calogero particles,” J. Phys. A **39** (2006) 12793 [hep-th/0607033].
- [24] R. Bhattacharyya, R. de Mello Koch and M. Stephanou, “Exact Multi-Restricted Schur Polynomial Correlators,” JHEP **0806** (2008) 101 [arXiv:0805.3025 [hep-th]].
- [25] S. Collins, “Restricted Schur Polynomials and Finite N Counting,” Phys. Rev. D **79** (2009) 026002 [arXiv:0810.4217 [hep-th]].
- [26] V. Balasubramanian, D. Berenstein, B. Feng and M. -x. Huang, “D-branes in Yang-Mills theory and emergent gauge symmetry,” JHEP **0503** (2005) 006 [hep-th/0411205].
- [27] R. de Mello Koch, J. Smolic and M. Smolic, “Giant Gravitons - with Strings Attached (I),” JHEP **0706** (2007) 074 [hep-th/0701066]; “Giant Gravitons - with Strings Attached (II),” JHEP **0709** (2007) 049 [hep-th/0701067].
- [28] D. Bekker, R. de Mello Koch and M. Stephanou, “Giant Gravitons - with Strings Attached. III.,” JHEP **0802** (2008) 029 [arXiv:0710.5372 [hep-th]].
- [29] Y. Kimura and H. Lin, “Young diagrams, Brauer algebras, and bubbling geometries,” JHEP **1201** (2012) 121 [arXiv:1109.2585 [hep-th]].
- [30] K. Koike, “On the Decomposition of Tensor Products of the Representations of the Classical Groups: By Means of the Universal Characters,” Advances in Mathematics **74** (1989) 57-86.
- [31] T. Halverson, “Characters of the centralizer algebras of mixed tensor representations of $Gl(r, C)$ and the quantum group $U_q(gl(r, C))$,” Pacific J. Math. Volume 174, No. 2 (1996) 359.
- [32] K. Okuyama, “N=4 SYM on $R \times S^{*3}$ and PP wave,” JHEP **0211** (2002) 043 [hep-th/0207067].
- [33] N. Kim, T. Klose and J. Plefka, “Plane wave matrix theory from N=4 superYang-Mills on $R \times S^{*3}$,” Nucl. Phys. B **671** (2003) 359 [hep-th/0306054].
- [34] G. Ishiki, Y. Takayama and A. Tsuchiya, “N=4 SYM on $R \times S^{*3}$ and theories with 16 supercharges,” JHEP **0610** (2006) 007 [hep-th/0605163].
- [35] K. Skenderis and M. Taylor, “Anatomy of bubbling solutions,” JHEP **0709** (2007) 019 [arXiv:0706.0216 [hep-th]]; “Kaluza-Klein holography,” JHEP **0605** (2006) 057 [hep-th/0603016].
- [36] R. d. M. Koch, N. Ives and M. Stephanou, “Correlators in Nontrivial Backgrounds,” Phys. Rev. D **79** (2009) 026004 [arXiv:0810.4041 [hep-th]].
- [37] R. de Mello Koch, T. K. Dey, N. Ives and M. Stephanou, “Correlators Of Operators with a Large R-charge,” JHEP **0908** (2009) 083 [arXiv:0905.2273 [hep-th]]; “Hints of Integrability Beyond the Planar Limit: Nontrivial Backgrounds,” JHEP **1001** (2010) 014 [arXiv:0911.0967 [hep-th]].

- [38] T. W. Brown, “Cut-and-join operators and N=4 super Yang-Mills,” JHEP **1005** (2010) 058 [arXiv:1002.2099 [hep-th]].
- [39] J. Pasukonis and S. Ramgoolam, “From counting to construction of BPS states in N=4 SYM,” JHEP **1102** (2011) 078 [arXiv:1010.1683 [hep-th]].
- [40] R. d. M. Koch, G. Mashile and N. Park, “Emergent Threebrane Lattices,” Phys. Rev. D **81** (2010) 106009 [arXiv:1004.1108 [hep-th]].
- [41] V. De Comarmond, R. de Mello Koch and K. Jefferies, “Surprisingly Simple Spectra,” JHEP **1102** (2011) 006 [arXiv:1012.3884 [hep-th]].
- [42] W. Carlson, R. d. M. Koch and H. Lin, “Nonplanar Integrability,” JHEP **1103** (2011) 105 [arXiv:1101.5404 [hep-th]].
- [43] R. d. M. Koch, B. A. E. Mohammed and S. Smith, “Nonplanar Integrability: Beyond the SU(2) Sector,” arXiv:1106.2483 [hep-th].
- [44] R. d. M. Koch, M. Dessein, D. Giataganas and C. Mathwin, “Giant Graviton Oscillators,” JHEP **1110** (2011) 009 [arXiv:1108.2761 [hep-th]].
- [45] R. de Mello Koch, G. Kemp and S. Smith, “From Large N Nonplanar Anomalous Dimensions to Open Spring Theory,” Phys. Lett. B **711** (2012) 398 [arXiv:1111.1058 [hep-th]].
- [46] R. de Mello Koch, P. Diaz and H. Soltanpanahi, “Non-planar Anomalous Dimensions in the sl(2) Sector,” arXiv:1111.6385 [hep-th].
- [47] R. d. M. Koch and S. Ramgoolam, “A double coset ansatz for integrability in AdS/CFT,” arXiv:1204.2153 [hep-th].
- [48] R. d. M. Koch, G. Kemp, B. A. E. Mohammed and S. Smith, “Nonplanar integrability at two loops,” arXiv:1206.0813 [hep-th].